

A CLASS OF SYMMETRIC GRAPHS WITH 2-ARC-TRANSITIVE QUOTIENTS

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ABSTRACT. Let Γ be a finite X -symmetric graph with a nontrivial X -invariant partition \mathcal{B} on $V(\Gamma)$ such that $\Gamma_{\mathcal{B}}$ is a connected $(X, 2)$ -arc-transitive graph and Γ is not a multicover of $\Gamma_{\mathcal{B}}$. A characterization of (Γ, X, \mathcal{B}) was given in [20] for the case where $|\Gamma(C) \cap B| = 2$ for $B \in \mathcal{B}$ and $C \in \Gamma_{\mathcal{B}}(B)$. This motivates us to investigate the case where $|\Gamma(C) \cap B| = 3$, that is, $\Gamma[B, C]$ is isomorphic to one of $3K_2$, $K_{3,3} - 3K_2$ and $K_{3,3}$. This investigation requires a study on $(X, 2)$ -arc-transitive graphs of valency 4 or 7. Based on the results in [14], we give a characterization of tetravalent $(X, 2)$ -arc-transitive graphs; and as a byproduct, we prove that every tetravalent $(X, 2)$ -transitive graph is either the complete graph on 5 vertices or a near n -gonal graph for some $n \geq 4$. We show that a heptavalent $(X, 2)$ -arc-transitive graph Σ can occur as $\Gamma_{\mathcal{B}}$ if and only if $X_{\tau}^{\Sigma(\tau)} \cong PSL(3, 2)$ for $\tau \in V(\Sigma)$.

KEYWORDS. Symmetric graph, quotient graph, three-arc graph, double star graph, near n -gonal graph.

1. INTRODUCTION

In this paper, all graphs are assumed to be finite, nonempty, simple and undirected. This paper involves graphs, permutation groups and designs, the reader is referred to [3], [4] and [2] respectively for the notation and terminology not mentioned here.

Let Σ be a regular graph with vertex set $V(\Sigma)$ and edge set $E(\Sigma)$. By $val(\Sigma)$ we denote the valency of Σ . For an integer $s \geq 1$ and an $(s+1)$ -sequence $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_s)$ of $V(\Sigma)$, set $\alpha^{-1} := (\alpha_s, \alpha_{s-1}, \dots, \alpha_0)$, α is called an s -arc of Σ if $\{\alpha_i, \alpha_{i+1}\} \in E(\Sigma)$ for $i = 0, 1, \dots, s-1$, and $\alpha_{i-1} \neq \alpha_{i+1}$ for $i = 1, 2, \dots, s-1$. An s -arc $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_s)$ is called an s -dipath if $\alpha_i \neq \alpha_j$ for $i, j \in \{0, 1, \dots, s\}$ with $i \neq j$. Evidently, α is an s -arc (s -dipath, respectively) of Σ if and only if α^{-1} is an s -arc (s -dipath, respectively) of Σ . For any s -dipath $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_s)$ of Σ , identifying α and α^{-1} gives rise to an s -path $[\alpha_0, \alpha_1, \dots, \alpha_s]$ of Σ . Denote by $Arc_s(\Sigma)$ ($Path_s(\Sigma)$, respectively) the set of s -arcs (s -paths) of Σ . In the case where $s = 1$, we use *arc* and $Arc(\Sigma)$ in place of 1-arc and $Arc_1(\Sigma)$, respectively.

This work is supported by the 973 Program and the NSF of China.

2000 *Mathematics Subject Classification* 05C25.

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Let X be a group acting on $V(\Sigma)$. The induced action of X on $V(\Sigma) \times V(\Sigma)$ is defined by $(\tau, \sigma)^x = (\tau^x, \sigma^x)$ for $(\tau, \sigma) \in V(\Sigma) \times V(\Sigma)$ and $x \in X$. We say that X preserves the adjacency of Σ if $\text{Arc}(\Sigma)^x = \text{Arc}(\Sigma)$, for all $x \in X$. The graph Σ is said to be X -vertex-transitive if X preserves the adjacency of Σ and acts transitively on $V(\Sigma)$; and Σ is said to be (X, s) -arc-transitive ((X, s) -arc-regular, respectively) if in addition the induced action of X on $\text{Arc}_s(\Sigma)$ is transitive (regular, respectively). Further, Σ is said to be (X, s) -transitive if Σ is (X, s) -arc-transitive but is not $(X, s+1)$ -arc-transitive. An $(X, 1)$ -arc-transitive graph is usually called an X -symmetric graph. For $\tau \in V(\Sigma)$, we denote by X_τ the point-stabilizer of τ in X . It is well-known that, for $s \in \{1, 2\}$, an X -vertex-transitive graph Σ is (X, s) -arc-transitive if and only if X_τ is s -transitive on the neighborhood $\Sigma(\tau) := \{\sigma \in V(\Sigma) \mid (\tau, \sigma) \in \text{Arc}(\Sigma)\}$ of τ in Σ . The reader is referred to [1] for basic results about symmetric graphs.

Let Γ be a finite X -symmetric graph admits a nontrivial X -invariant partition \mathcal{B} on $V(\Gamma)$, that is, $1 < |\mathcal{B}| < V(\Gamma)$ and $B^x := \{\mathbf{v}^x \mid \mathbf{v} \in B\} \in \mathcal{B}$ for $B \in \mathcal{B}$ and $x \in X$. (Such a graph is said to be an *imprimitive* X -symmetric graph.) The *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ with respect to \mathcal{B} is defined to be the graph with vertex set \mathcal{B} such that, for $B, C \in \mathcal{B}$, B is adjacent to C in $\Gamma_{\mathcal{B}}$ if and only if there exists some $\mathbf{v} \in B$ adjacent to some $\mathbf{u} \in C$ in Γ . It is easy to see that X acts transitively on the vertex set and on the arc set of $\Gamma_{\mathcal{B}}$, that is, $\Gamma_{\mathcal{B}}$ is X -symmetric. We always assume that $\Gamma_{\mathcal{B}}$ has at least one edge, which implies that each $B \in \mathcal{B}$ is an independent set of $V(\Gamma)$.

It has been observed in the literature that the quotient graphs of an $(X, 2)$ -arc-transitive graph are usually not $(X, 2)$ -arc-transitive, and that an X -symmetric graph with an $(X, 2)$ -arc-transitive quotient itself is not necessarily $(X, 2)$ -arc-transitive. (For example, several examples are given in [5, 6] for the first situation; and for the second situation, it is shown in [14] that every connected $(X, 3)$ -arc-transitive graph is a quotient graph of at least one X -symmetric graph which is not $(X, 2)$ -arc-transitive.) This observation gave rise to a series of intensively studies of the following two questions (Q1) and (Q2) [20, 10] by investigating ‘local’ structures of imprimitive symmetric graphs and their quotient graphs.

- (Q1) When can $\Gamma_{\mathcal{B}}$ be $(X, 2)$ -arc-transitive?
- (Q2) What information of the structure of Γ can we obtain from an $(X, 2)$ -arc-transitive quotient $\Gamma_{\mathcal{B}}$ of Γ ?

For $B \in \mathcal{B}$ and $\mathbf{v} \in V(\Gamma)$, we set $\Gamma(B) := \bigcup_{\mathbf{u} \in B} \Gamma(\mathbf{u})$, $\Gamma_{\mathcal{B}}(B) := \{C \in \mathcal{B} \mid (B, C) \in \text{Arc}(\Gamma_{\mathcal{B}})\}$ and $\Gamma_{\mathcal{B}}(\mathbf{v}) := \{C \in \mathcal{B} \mid \mathbf{v} \in \Gamma(C)\}$. Let $\mathcal{D}(B) := (B, \Gamma_{\mathcal{B}}(B), |)$ denote the incidence structure such that $\mathbf{v}|C$ for $\mathbf{v} \in B$, $C \in \Gamma_{\mathcal{B}}(B)$ if and only if $C \in \Gamma_{\mathcal{B}}(\mathbf{v})$. For any $B \in \mathcal{B}$, $C \in \Gamma_{\mathcal{B}}(B)$ and $\mathbf{v} \in B$, as Γ is X -symmetric, $v := |B|$, $k := |\Gamma(C) \cap B|$, $r := |\Gamma_{\mathcal{B}}(\mathbf{v})|$ and $b := \text{val}(\Gamma_{\mathcal{B}})$ are independent of the choice of B and \mathbf{v} , and $\mathcal{D}(B)$ is an X_B -flag-transitive 1- (v, k, r) design with b blocks [14, Lemma 2.1]. Γ is said to be a multicover of

$\Gamma_{\mathcal{B}}$ if $k = v$. For $(B, C) \in \text{Arc}(\Gamma_{\mathcal{B}})$, denote by $\Gamma[B, C]$ the bipartite subgraph of Γ induced by $(\Gamma(C) \cap B) \cup (\Gamma(B) \cap C)$. Then $\Gamma[B, C]$ is independent of the choice of $(B, C) \in \text{Arc}(\Gamma_{\mathcal{B}})$ up to isomorphism, and $X_B \cap X_C$ acts transitively on the edges of $\Gamma[B, C]$.

Without doubt, the triple $(\Gamma_{\mathcal{B}}, \Gamma[B, C], \mathcal{D}(B))$ mirrors ‘global’ and ‘local’ information of the structure of Γ , which allows us to reconstruct Γ in some sense. This approach to imprimitive symmetric graphs have received attention in the literature. Gardiner and Praeger [7] suggested to analyse these three configurations, $(\Gamma, \Gamma[B, C], \mathcal{D}(B))$, and they discussed the case when Γ is X -locally primitive, that is, the stabilizer of a vertex $\mathbf{v} \in V(\Gamma)$ in X acts primitively on the neighbourhood $\Gamma(\mathbf{v})$ of the vertex in Γ . In [8, 9] they considered the case when the quotient $\Gamma_{\mathcal{B}}$ is a complete graph and the setwise stabiliser X_B (the subgroup of X fixing B setwise) is 2-transitive on B . Li, Praeger and Zhou [12] proved that, if $k = v - 1 \geq 2$, then $\mathcal{D}(B)$ contains no repeated blocks (that is, the subsets of B incident with distinct blocks of $\mathcal{D}(B)$ are distinct) if and only if $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive, and further they found an elegant construction (called the 3-arc graph construction) for constructing all such graphs from $\Gamma_{\mathcal{B}}$. Iranmanesh, Praeger and Zhou [10], Lu and Zhou [14] studied the case where the quotient $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive and obtained a series of interesting results. In particular, Lu and Zhou [14] found the second type 3-arc graph construction, which led to a classification [21] of a family of symmetric graphs. The reader is referred to [16, 17, 18, 19, 20] for further more developments in this topic.

In answering the above two questions, a relatively explicit classification of (Γ, X, \mathcal{B}) has been given in [20], when $\Gamma_{\mathcal{B}}$ is a connected $(X, 2)$ -arc-transitive graph such that $2 = k \leq v - 1$. This motivated us in this paper to investigate the case where $k = 3$, and then we give a characterization for this case.

For a group X acting on a set V and a subset B of V , denote by $X_{(B)}$ (X_B , respectively) the point-wise (set-wise, respectively) stabilizer of B in X , and by X_B^B the permutation group induced by X_B on B . Then $X_B^B \cong X_B / X_{(B)}$.

The following is a summary of the main results of this paper, which is a sketch of our answer to (Q1) and (Q2) in the case where $k = 3$. More details will be given in Theorem 4.1.

Theorem 1.1. *Let Γ be an X -symmetric graph with an X -invariant partition \mathcal{B} on $V(\Gamma)$ such that $k = 3$ and $\text{val}(\Gamma_{\mathcal{B}}) \geq 2$. Let $B \in \mathcal{B}$. If X is faithful on $V(\Gamma)$ and Γ is not a multicover of $\Gamma_{\mathcal{B}}$, then $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive if and only if one of the following cases occurs.*

- (a) $(v, b, r) = (4, 4, 3)$ and $X_B^B \cong A_4$ or S_4 ;
- (b) $(v, b, r) = (6, 4, 2)$ and $X_B^B \cong A_4$ or S_4 ;
- (c) $(v, b, r) = (7, 7, 3)$ and $X_B^B \cong \text{PSL}(3, 2)$;
- (d) $v = 3b \geq 6$, $r = 1$ and X_B acts 2-transitively on the blocks of $\mathcal{D}(B)$.

2. GRAPHS CONSTRUCTED FROM GIVEN GRAPHS

In this section, we aim to restate several graphs constructed from given graphs, as well as some of their properties, which turn out to be useful in a further characterization of (Γ, X, \mathcal{B}) stated in Theorem 1.1. Hereafter, we denote by \mathcal{G} the set of triples (Γ, X, \mathcal{B}) such that Γ is a finite X -symmetric graph with a nontrivial X -invariant partition \mathcal{B} on $V(\Gamma)$, $\text{val}(\Gamma_{\mathcal{B}}) \geq 2$ and Γ is not a multicover of $\Gamma_{\mathcal{B}}$, and by $\hat{\mathcal{G}}$ the subset of \mathcal{G} such that $\Gamma_{\mathcal{B}}$ is connected and X acts faithfully on $V(\Gamma)$, that is, $\cap_{v \in V(\Gamma)} X_v = 1$.

The following two propositions are quoted from [14].

Proposition 2.1. *Let Σ be a finite $(X, 2)$ -arc-transitive graph with $\text{val}(\Sigma) \geq 2$. Let Δ be a self-paired subset of $\text{Arc}_3(\Sigma)$, that is, $\alpha^{-1} \in \Delta$ whenever $\alpha \in \Delta$. Define $\mathfrak{J} := \mathfrak{J}(\Sigma, \Delta)$ to be the graph with vertex set $\text{Path}_2(\Sigma)$ and edge set $\{[\alpha_0, \alpha_1, \alpha_2], [\alpha_1, \alpha_2, \alpha_3]\} \mid (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \Delta\}$. Set $P_\tau := \{[\tau_1, \tau, \tau_2] \in \text{Path}_2(\Sigma) \mid \tau_1, \tau_2 \in \Sigma(\tau)\}$ for $\tau \in V(\Sigma)$, and $\mathcal{P} := \{P_\sigma \mid \sigma \in V(\Sigma)\}$. If Δ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$, then $(\mathfrak{J}, X, \mathcal{P}) \in \mathcal{G}$ and $\Sigma \cong \mathfrak{J}_{\mathcal{P}}$.*

The following lemma improves [14, Theorem 4.10].

Lemma 2.2. *Let $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$ with $b \geq 3$ and $r = 2$. Set*

$$\Delta := \left\{ (C, B(v), B(u), D) \mid \begin{array}{l} (v, u) \in \text{Arc}(\Gamma) \\ v \in B(v) \in \mathcal{B}, u \in B(u) \in \mathcal{B} \\ C \in \Gamma_{\mathcal{B}}(v), D \in \Gamma_{\mathcal{B}}(u), C \neq B(u), D \neq B(v) \end{array} \right\}.$$

Suppose that $|\Gamma(D) \cap B_0 \cap \Gamma(C)| \neq 0$ for any 2-path $[D, B_0, C]$ of $\Gamma_{\mathcal{B}}$ with a given middle vertex $B_0 \in \mathcal{B}$. Then $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive and $\lambda := |\Gamma(D) \cap B_0 \cap \Gamma(C)|$ is independent of the choices of $[D, B_0, C]$ and B_0 ; further, Δ is a self-paired X -orbit on $\text{Arc}_3(\Gamma_{\mathcal{B}})$, and either

- (a) $\lambda = 1$ and $\Gamma \cong \mathfrak{J}(\Gamma_{\mathcal{B}}, \Delta)$; or
- (b) $\lambda \geq 2$ and Γ admits a second nontrivial X -invariant partition

$$\mathcal{Q} := \{\Gamma(D) \cap B \cap \Gamma(C) \mid [D, B, C] \in \text{Path}_2(\Gamma_{\mathcal{B}})\}$$

on $V(\Gamma)$, which is a proper refinement of \mathcal{B} such that $\Gamma_{\mathcal{Q}} \cong \mathfrak{J}(\Gamma_{\mathcal{B}}, \Delta)$.

Proof. Note that $b \geq 3$. Take three distinct blocks $C, D, E \in \Gamma_{\mathcal{B}}(B_0)$. Since $|\Gamma(D) \cap B_0 \cap \Gamma(C)| \neq 0$ and $|\Gamma(E) \cap B_0 \cap \Gamma(C)| \neq 0$, there exist $v, u \in \Gamma(C) \cap B_0$ with $v \in \Gamma(D)$ and $u \in \Gamma(E)$. Let $v', u' \in C$ be such that $(v, v'), (u, u') \in \text{Arc}(\Gamma)$. Then $(v, v')^x = (u, u')$ for some $x \in X$ as Γ is X -symmetric. So $v^x = u$ and $v'^x = u'$, it implies $B_0^x = B_0$ and $C^x = C$, hence $x \in X_{B_0} \cap X_C$. Further $C, D^x, E \in \Gamma_{\mathcal{B}}(u)$, it follows that $D^x = E$ as $r := |\Gamma_{\mathcal{B}}(u)| = 2$. Thus $X_{B_0} \cap X_C$ is transitive on $\Gamma_{\mathcal{B}}(B_0) \setminus \{C\}$, it follows that X_{B_0} is 2-transitive on $\Gamma_{\mathcal{B}}(B_0)$. Therefore, $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive. Then, by [14], $\lambda \geq 1$ is a constant number; and if $\lambda = 1$, Δ is a self-paired X -orbit on $\text{Arc}_3(\Gamma_{\mathcal{B}})$ and $\Gamma \cong \mathfrak{J}(\Gamma_{\mathcal{B}}, \Delta)$. In the following we assume $\lambda \geq 2$.

We first show \mathcal{Q} is an X -invariant partition of $V(\Gamma)$. Take two arbitrary 2-paths $[D_1, B_1, C_1]$ and $[D_2, B_2, C_2]$ of $\Gamma_{\mathcal{B}}$. Suppose that there exists some $\mathbf{v} \in V(\Gamma)$ such that $\mathbf{v} \in (\Gamma(D_1) \cap B_1 \cap \Gamma(C_1)) \cap (\Gamma(D_2) \cap B_2 \cap \Gamma(C_2))$. Then $B_1 = B_2$ and $C_i, D_i \in \Gamma_{\mathcal{B}}(\mathbf{v})$ for $i = 1, 2$. Since $r = 2$, we have that $\{C_1, D_1\} = \{C_2, D_2\}$, thus $[D_1, B_1, C_1] = [D_2, B_2, C_2]$. It follows that \mathcal{Q} is a partition of $V(\Gamma)$. For any $[D, B, C] \in \text{Path}_2(\Gamma_{\mathcal{B}})$ and $x \in X$, we have $[D, B, C]^x = [D^x, B^x, C^x] \in \text{Path}_2(\Gamma_{\mathcal{B}})$ and so $(\Gamma(D) \cap B \cap \Gamma(C))^x = \Gamma(D^x) \cap B^x \cap \Gamma(C^x) \in \mathcal{Q}$. Thus \mathcal{Q} is X -invariant. Noting that Γ is not a multicover of $\Gamma_{\mathcal{B}}$, we know $|B| > |\Gamma(D) \cap B \cap \Gamma(C)| := \lambda \geq 2$, so \mathcal{Q} is a proper refinement of \mathcal{B} . In particular, the pair $(\mathcal{B}, \mathcal{Q})$ gives an X -invariant partition $\tilde{\mathcal{B}}$ of $V(\Gamma_{\mathcal{Q}})$.

Consider the quotient graph $(\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}$ of $\Gamma_{\mathcal{Q}}$ with respect to $\tilde{\mathcal{B}}$. For any 2-path $[\bar{D}, \bar{B}, \bar{C}]$ of $(\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}$ and any $\bar{\mathbf{v}} \in V(\Gamma_{\mathcal{Q}})$, we have $|(\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}(\bar{\mathbf{v}})| = 2$ and $|\Gamma_{\mathcal{Q}}(\bar{D}) \cap \bar{B} \cap \Gamma_{\mathcal{Q}}(\bar{C})| = 1$. It follows from (a) that $\Gamma_{\mathcal{Q}} \cong \mathbb{J}((\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}, \bar{\Delta})$, where $\bar{\Delta} = \{(\bar{C}, \bar{B}(\bar{\mathbf{v}}), \bar{B}(\bar{\mathbf{u}}), \bar{D}) \mid (C, B(\mathbf{v}), B(\mathbf{u}), D) \in \Delta\}$. Moreover, it is easily shown that $\tilde{\mathcal{B}} \rightarrow \mathcal{B}$, $\bar{B} \mapsto B$ is an isomorphism from $(\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}$ to $\Gamma_{\mathcal{B}}$. Therefore, $\Gamma_{\mathcal{Q}} \cong \mathbb{J}((\Gamma_{\mathcal{Q}})_{\tilde{\mathcal{B}}}, \bar{\Delta}) \cong \mathbb{J}(\Gamma_{\mathcal{B}}, \Delta)$. \blacksquare

For a finite X -symmetric graph Σ with valency no less than three, let $J(\Sigma)$ be the set of pairs $([\tau_1, \tau, \tau_2], [\sigma_1, \sigma, \sigma_2])$ of 2-paths of Σ such that $\sigma \in \Sigma(\tau) \setminus \{\tau_1, \tau_2\}$, $\tau \in \Sigma(\sigma) \setminus \{\sigma_1, \sigma_2\}$. A subset Λ of $J(\Sigma)$ is said to be self-paired if $([\tau_1, \tau, \tau_2], [\sigma_1, \sigma, \sigma_2]) \in \Lambda$ always implies that $([\sigma_1, \sigma, \sigma_2], [\tau_1, \tau, \tau_2]) \in \Lambda$.

Proposition 2.3. *Let Σ be a finite $(X, 2)$ -arc-transitive graph with $\text{val}(\Sigma) \geq 3$ and let Λ be a self-paired X -orbit on $J(\Sigma)$. Define a graph $\Psi := \Psi(\Sigma, \Lambda)$ with vertex set $\text{Path}_2(\Sigma)$ such that two 2-paths $\mathfrak{r}, \mathfrak{s}$ are adjacent if and only if $(\mathfrak{r}, \mathfrak{s}) \in \Lambda$. Then Ψ is X -symmetric and \mathcal{P} is a nontrivial X -invariant partition of $V(\Psi)$ with $\Sigma \cong \Psi_{\mathcal{P}}$, where \mathcal{P} is defined as in Proposition 2.1.*

We now quote a result about 3-arc graphs [12].

Proposition 2.4. *Let Σ be a finite $(X, 2)$ -arc-transitive graph with $\text{val}(\Sigma) \geq 3$ and let Δ be a self-paired X -orbit on $\text{Arc}_3(\Sigma)$. The 3-arc graph $\Xi := \Xi(\Sigma, \Delta)$ with respect to Δ is defined to be the graph with vertex set $\text{Arc}(\Sigma)$ such that two arcs (τ, τ_1) and (σ, σ_1) of Σ are adjacent in Ξ if and only if $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$. Then $(\Xi, X, \mathcal{A}) \in \mathcal{G}$ and $\Sigma \cong \Xi_{\mathcal{A}}$, where $\mathcal{A} := \{A_{\tau} \mid \tau \in V(\Sigma)\}$ and $A_{\tau} := \{(\tau, \sigma) \mid \sigma \in \Sigma(\tau)\}$ for $\tau \in V(\Sigma)$.*

Lemma 2.5. *Let Σ , X , Δ and Ξ be as in Proposition 2.4. Then $r_{\mathcal{A}} := |\Xi_{\mathcal{A}}((\tau, \tau_1))| = \text{val}(\Sigma) - 1$ and $\text{val}(\Xi) = r_{\mathcal{A}}\ell$, where $(\tau, \tau_1), (\tau, \sigma) \in V(\Xi) = \text{Arc}(\Sigma)$ and ℓ is the valency of $\Xi[A_{\tau}, A_{\sigma}]$.*

Proof. For any arc (τ, σ) of Σ , there is a 3-arc $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$ as X acts transitively on arcs of Σ . Then A_{τ} and A_{σ} are adjacent in $\Xi_{\mathcal{A}}$. It implies that $\text{val}(\Xi) = r_{\mathcal{A}}\ell$. So it suffices to show $r_{\mathcal{A}} = \text{val}(\Sigma) - 1$. Let $(\sigma', \sigma'_1) \in \text{Arc}(\Sigma)$. Note that Δ is self-paired. Then $\{(\tau, \tau_1), (\sigma', \sigma'_1)\} \in E(\Xi)$ if and only if $(\tau_1, \tau, \sigma', \sigma'_1) \in \Delta$. In particular, if $A_{\sigma'} \in \Xi_{\mathcal{A}}((\tau, \tau_1))$ then

$\tau_1 \neq \sigma'$ and $(\tau, \sigma') \in \text{Arc}(\Sigma)$. Then σ' , and hence $A_{\sigma'}$, has at most $\text{val}(\Sigma) - 1$ choices. So $r_{\mathcal{A}} \leq \text{val}(\Sigma) - 1$. On the other hand, since Σ is $(X, 2)$ -arc-transitive, for any $\sigma' \in \Sigma(\tau)$ with $\sigma' \neq \tau_1$, there is some $x \in X$ such that $(\tau_1, \tau, \sigma, \sigma_1)^x = (\tau_1, \tau, \sigma', \sigma_1^x) \in \Delta$. It follows that $\{(\tau, \tau_1), (\sigma', \sigma_1^x)\} \in E(\Xi)$, and so $A_{\sigma'} \in \Xi_{\mathcal{A}}((\tau, \tau_1))$. Then $r_{\mathcal{A}} \geq \text{val}(\Sigma) - 1$. Thus $r_{\mathcal{A}} = \text{val}(\Sigma) - 1$. ■

3. DOUBLE STAR GRAPHS

If $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$ such that $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive then, by [14], Γ or a quotient of Γ is isomorphic to one of $|E(\Gamma_{\mathcal{B}})|K_2$, $\mathfrak{I}(\Gamma_{\mathcal{B}}, \Delta)$, $\Psi(\Gamma_{\mathcal{B}}, \Lambda)$ and $\Xi(\Gamma_{\mathcal{B}}, \Delta)$ for $r = 1, 2, b-2$ and $b-1$, respectively, where Δ is a self-paired X -orbit on $\text{Arc}_3(\Gamma_{\mathcal{B}})$ and Λ is a self-paired X -orbit on $J(\Gamma_{\mathcal{B}})$. This motivates us in this section to consider the general case where $1 \leq r \leq b-1$, and introduce the stars and the double stars for a given graph. We shall show that there is a close connection between Γ and the graph constructed from a certain set of double stars of $\Sigma := \Gamma_{\mathcal{B}}$.

3.1. Stars of symmetric graphs. Let Σ be an X -symmetric graph with valency no less than 2. For $\tau \in V(\Sigma)$ and an \mathbb{k} -subset S of $\Sigma(\tau)$, we call $\mathfrak{s}(\tau, S) := \{(\tau, \sigma) \in \text{Arc}(\Sigma) \mid \sigma \in S\}$ a \mathbb{k} -star of Σ with respect to τ and S . Set $\mathcal{S}t_{\tau}^{\mathbb{k}}(\Sigma) := \{\mathfrak{s}(\tau, S) \mid S \subseteq \Sigma(\tau), |S| = \mathbb{k}\}$ and $\mathcal{S}t^{\mathbb{k}}(\Sigma) := \cup_{\tau \in V(\Sigma)} \mathcal{S}t_{\tau}^{\mathbb{k}}(\Sigma)$. A star $\mathfrak{s} := \mathfrak{s}(\tau, S)$ is said to be $X_{\mathfrak{s}}$ -symmetric if $X_{\tau} \cap X_S$ acts transitively on S . A nonempty subset \mathcal{S} of $\mathcal{S}t^{\mathbb{k}}(\Sigma)$ is said to be X -symmetric if \mathcal{S} is X -transitive and \mathfrak{s} is $X_{\mathfrak{s}}$ -symmetric for some $\mathfrak{s} \in \mathcal{S}$.

Let \mathcal{S} be an X -symmetric subset of $\mathcal{S}t^{\mathbb{k}}(\Sigma)$. For $\tau \in V(\Sigma)$, set $\mathcal{S}_{\tau} = \{\mathfrak{s} \in \mathcal{S} \mid \mathfrak{s} = \mathfrak{s}(\tau, S), S \subseteq \Sigma(\tau), |S| = \mathbb{k}\}$. Define an incidence structure $\mathbb{D}(\tau) := (\Sigma(\tau), \mathcal{S}_{\tau}, \parallel)$ in which $\sigma \parallel \mathfrak{s}(\tau, S)$, for $\sigma \in \Sigma(\tau)$, $\mathfrak{s}(\tau, S) \in \mathcal{S}_{\tau}$, if and only if $\sigma \in S$. A pair (σ, \mathfrak{s}) with $\sigma \parallel \mathfrak{s}$ is said to be a *flag* of $\mathbb{D}(\tau)$. Let $\mathfrak{r} := |\{\mathfrak{s}(\tau, S) \in \mathcal{S}_{\tau} \mid \sigma \in S\}|$, $\mathfrak{b} := |\mathcal{S}_{\tau}|$ and $\mathfrak{v} := \text{val}(\Sigma)$. Then it is easy to see that $\mathbb{D}(\tau)$ is an X_{τ} -flag-transitive $1-(\mathfrak{v}, \mathfrak{k}, \mathfrak{r})$ design with \mathfrak{b} blocks. Moreover, $\mathbb{D}(\tau)$ is independent of the choice of $\tau \in V(\Sigma)$ up to isomorphism.

The following Lemma 3.1 says that, for $\tau \in V(\Sigma)$, an arbitrary X_{τ} -flag-transitive $1-(\mathfrak{v}, \mathfrak{k}, \mathfrak{r})$ design can be constructed as above in some sense. Let $\mathfrak{D}(\tau) := (\Sigma(\tau), \mathfrak{B}, \mathfrak{I})$ be an X_{τ} -flag-transitive $1-(\mathfrak{v}, \mathfrak{k}, \mathfrak{r})$ design. It may happen that distinct blocks \mathfrak{b}_1 and \mathfrak{b}_2 of $\mathfrak{D}(\tau)$ have the same trace $\{\sigma \mid \sigma \mathfrak{I} \mathfrak{b}_1\} = \{\sigma \mid \sigma \mathfrak{I} \mathfrak{b}_2\}$. Since $\mathfrak{D}(\tau)$ is flag-transitive, the number of blocks with the same trace is a constant, say $m(\mathfrak{D}(\tau))$, called the multiplicity of $\mathfrak{D}(\tau)$. Let $\mathfrak{D}'(\tau)$ be the design with vertex set $\Sigma(\tau)$ and blocks being the traces of blocks of $\mathfrak{D}(\tau)$. Then $\mathfrak{D}'(\tau)$ is an X_{τ} -flag-transitive $1-(\mathfrak{v}, \mathfrak{k}, \mathfrak{r}')$ design, where $\mathfrak{r}' = \frac{\mathfrak{r}}{m(\mathfrak{D}(\tau))}$.

Lemma 3.1. *Let Σ be an X -symmetric graph with valency $\mathfrak{v} \geq 2$ and $\mathfrak{D}(\tau)$ be an X_{τ} -flag-transitive $1-(\mathfrak{v}, \mathfrak{k}, \mathfrak{r})$ design with \mathfrak{b} blocks, where $1 \leq \mathfrak{k} \leq \mathfrak{v} - 1$ and $\tau \in V(\Sigma)$. Set $\mathcal{S} := \{\mathfrak{s}(\tau^x, S^x) \mid x \in X, S \in \mathfrak{D}'(\tau)\}$. Then \mathcal{S} is*

X -symmetric, and $\mathfrak{D}'(\tau) \cong \mathbb{D}(\tau)$ is an X_τ -flag-transitive $1-(v, k, \frac{r}{m(\mathfrak{D}(\tau))})$ design with $\frac{b}{m(\mathfrak{D}(\tau))}$ blocks.

3.2. Double stars. Let L and R be k -subsets of $\Sigma(\tau)$ and $\Sigma(\sigma)$ respectively, set $\mathfrak{l} = \mathfrak{s}(\tau, L)$ and $\mathfrak{r} = \mathfrak{s}(\sigma, R)$, the pair $(\mathfrak{l}, \mathfrak{r})$ is called a k -double star of Σ if $\sigma \in L$ and $\tau \in R$. Denote by $DSt^k(\Sigma)$ the set of k -double stars of Σ . A nonempty subset Θ of $DSt^k(\Sigma)$ is said to be X -symmetric if $St(\Theta) := \{\mathfrak{l}, \mathfrak{r} \mid (\mathfrak{l}, \mathfrak{r}) \in \Theta\}$ is X -symmetric; and is self-paired if $(\mathfrak{l}, \mathfrak{r}) \in \Theta$ always implies that $(\mathfrak{r}, \mathfrak{l}) \in \Theta$.

Here we give a straightforward lemma by ignoring the proof.

Lemma 3.2. *Let Σ be an X -symmetric graph with valency $v \geq 2$ and k an integer with $1 \leq k \leq v$.*

- (a) *If \mathcal{S} is an X -symmetric orbit on $St^k(\Sigma)$, then for $\mathfrak{l} = \mathfrak{s}(\tau, L), \mathfrak{r} = \mathfrak{s}(\sigma, R) \in \mathcal{S}$ with $\sigma \in L$ and $\tau \in R$, $\Theta := \{(\mathfrak{l}^x, \mathfrak{r}^x) \mid x \in X\}$ is an X -symmetric orbit on $DSt^k(\Sigma)$ and $St(\Theta) = \mathcal{S}$.*
- (b) *Let Θ be an X -symmetric orbit on $DSt^k(\Sigma)$ and let $\tau, \sigma \in V(\Sigma)$. Then $(\tau, \sigma) \in Arc(\Sigma)$ if and only if there exist $\mathfrak{l} := \mathfrak{s}(\tau, L), \mathfrak{r} := \mathfrak{s}(\sigma, R) \in St(\Theta)$ such that $(\mathfrak{l}, \mathfrak{r}) \in \Theta$.*

The following example shows that an X -symmetric orbit Θ of k -double stars of an X -symmetric graph is not necessarily self-paired.

Example 3.3. Let Σ be a cubic $(X, 2)$ -arc-regular graph with a 3-arc $(\tau_1, \tau, \sigma, \sigma_1)$ such that there is no $x \in X$ maps this 3-arc into $(\sigma_1, \sigma, \tau, \tau_1)$. (See [1, 18c], for example.) Set $L = \{\tau_1, \sigma\}$, $R = \{\sigma_1, \tau\}$, $\mathfrak{l} := \mathfrak{s}(\tau, L)$ and $\mathfrak{r} = \mathfrak{s}(\sigma, R)$. Let $\Theta = \{(\mathfrak{l}^x, \mathfrak{r}^x) \mid x \in X\}$. Then Θ is an X -symmetric orbit on $DSt^2(\Sigma)$. However, it is easily shown that Θ is not self-paired.

Construction 3.4. *Let Σ be an X -symmetric graph with valency $v \geq 2$ and Θ be a self-paired X -symmetric orbit on $DSt^k(\Sigma)$ with $1 \leq k \leq v - 1$. Define a graph $\Pi(\Sigma, \Theta)$, called the double star graph of Σ with respect to Θ , with vertex set $St(\Theta)$ such that two k -stars \mathfrak{l} and \mathfrak{r} in $St(\Theta)$ are adjacent if and only if $(\mathfrak{l}, \mathfrak{r}) \in \Theta$.*

Theorem 3.5. *Let Σ , Θ and $\Gamma := \Pi(\Sigma, \Theta)$ be as in Construction 3.4. Set $\mathcal{S} = St(\Theta)$ and $\mathcal{B} = \{\mathcal{S}_\tau \mid \tau \in V(\Sigma)\}$, where $\mathcal{S}_\tau = \{\mathfrak{s} \in \mathcal{S} \mid \mathfrak{s} = \mathfrak{s}(\tau, S), S \subseteq \Sigma(\tau), |S| = k\}$. Then $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$ such that $\Gamma_{\mathcal{B}} \cong \Sigma$, and for $B = \mathcal{S}_\tau \in \mathcal{B}$, $\mathcal{D}(B) \cong \mathbb{D}^*(\tau)$, where $\mathbb{D}^*(\tau)$ is the dual design of $\mathbb{D}(\tau)$.*

Proof. It is easy to see that \mathcal{B} is an X -invariant partition of $V(\Gamma) = \mathcal{S}$. For any $\mathfrak{s} := \mathfrak{s}(\tau, S) \in B := \mathcal{S}_\tau \in \mathcal{B}$, as $1 \leq k = |S| \leq v - 1$, take $\sigma \in S$ and $\delta \in \Sigma(\tau) \setminus S$. Since Σ is X -symmetric, there exists $g \in X_\tau$ such that $\delta = \sigma^g$. Let $\mathfrak{l} = \mathfrak{s}^g$. Then $\mathfrak{s} \neq \mathfrak{l} \in \mathcal{S}_\tau$, thus $v = |\mathcal{S}_\tau| \geq 2$, and hence \mathcal{B} is a nontrivial X -invariant partition of $V(\Gamma)$. By Lemma 3.2, there exists $\mathfrak{r} \in \mathcal{S}_\delta$ such that $(\mathfrak{l}, \mathfrak{r}) \in \Theta$, hence $C := \mathcal{S}_\delta \in \Gamma_{\mathcal{B}}(B)$. If there exists $\mathfrak{r}' \in \mathcal{S}_\delta$

such that $(\mathfrak{s}, \mathfrak{r}') \in \Theta$, then $\delta \in S$, a contradiction. Thus $\mathfrak{s} \notin B \cap \Gamma(C)$ and $k = |B \cap \Gamma(C)| \leq v - 1$. Hence Γ is not a multicover of $\Gamma_{\mathcal{B}}$. It is easily shown by using Lemma 3.2 that Γ is X -symmetric, and $V(\Sigma) \rightarrow V(\Gamma_{\mathcal{B}})$, $\tau \mapsto \mathcal{S}_{\tau}$ is an isomorphism from Σ to $\Gamma_{\mathcal{B}}$.

For $\tau \in V(\Sigma)$ and $B = \mathcal{S}_{\tau}$, define a map $\pi : B \cup \Gamma_{\mathcal{B}}(B) \rightarrow \mathcal{S}_{\tau} \cup \Sigma(\tau)$; $\mathfrak{s}(\tau, S) \mapsto \mathfrak{s}(\tau, S)$, $C \mapsto \sigma$ for $\mathfrak{s}(\tau, S) \in B = \mathcal{S}_{\tau}$ and $C = \mathcal{S}_{\sigma} \in \Gamma_{\mathcal{B}}(B)$. Assume $C = \mathcal{S}_{\sigma} \in \Gamma_{\mathcal{B}}(B)$. Then by the definition of $\Gamma_{\mathcal{B}}$ and the construction of Γ there exist $\mathfrak{l} = \mathfrak{s}(\tau, L) \in B$ and $\mathfrak{r} = \mathfrak{s}(\sigma, R) \in C$ such that $(\mathfrak{l}, \mathfrak{r}) \in \Theta$. In particular, $\sigma \in L \subseteq \Sigma(\tau)$. Thus π is well-defined. Moreover π is a bijection. By the definition of $\mathcal{D}(B)$, for $\mathfrak{s} = \mathfrak{s}(\tau, S) \in B$ and $C = \mathcal{S}_{\sigma} \in \Gamma_{\mathcal{B}}(B)$, we know that $\mathfrak{s}|C$ if and only if there is some $\mathfrak{t} = \mathfrak{s}(\sigma, T) \in C$ such that $(\mathfrak{s}, \mathfrak{t}) \in \Theta$, that is, $\tau \in T$ and $\sigma \in S$; it follows that $\sigma \parallel \mathfrak{s}$. Now assume that $\sigma' \in \Sigma(\tau)$ and $\mathfrak{s}' = \mathfrak{s}(\tau, S')$ with $\sigma' \parallel \mathfrak{s}'$. Then $\sigma' \in S'$. Take some $\mathfrak{t}' = \mathfrak{s}(\tau', T')$ such that $(\mathfrak{s}', \mathfrak{t}') \in \Theta$. Then $\tau' \in S'$. Since \mathfrak{s}' is $X_{S'}$ -symmetric, there is some $x \in X_{\tau} \cap X_{S'}$ with $\tau'^x = \sigma'$. Thus $\mathfrak{s}'^x = \mathfrak{s}'$, $\mathfrak{t}'^x = \mathfrak{s}(\sigma', T'^x) \in \mathcal{S}_{\sigma'}$ and $(\mathfrak{s}', \mathfrak{t}'^x) = (\mathfrak{s}', \mathfrak{t}')^x \in \Theta$. Hence $\mathfrak{s}'|_{\mathcal{S}_{\sigma'}}$. The above argument says that π is an isomorphism from $\mathcal{D}(B)$ to $\mathbb{D}^*(\tau)$. So $\mathcal{D}(B) \cong \mathbb{D}^*(\tau)$. \blacksquare

Here we give the following sufficient condition which is useful in determining whether or not a double star graph exists.

Theorem 3.6. *Let Σ be an X -symmetric graph with valency $v \geq 2$ and let $\tau \in V(\Sigma)$. If there exists some X_{τ} -flag-transitive $1-(v, k, r)$ design $\mathcal{D}(\tau)$ on $\Sigma(\tau)$ for $1 \leq k \leq v - 1$ such that $\frac{r}{m(\mathcal{D}(\tau))}$ is odd, then there exists a self-paired X -symmetric orbit Θ on $DSt^k(\Sigma)$.*

Proof. By Lemma 3.1, setting $\mathcal{S} = \{\mathfrak{s}(\tau^x, S^x) \mid x \in X, S \in \mathcal{D}'(\tau)\}$, we know that $\mathcal{D}'(\tau) \cong \mathbb{D}(\tau)$ is an X_{τ} -flag-transitive $1-(v, k, \frac{r}{m(\mathcal{D}(\tau))})$ design with $\frac{b}{m(\mathcal{D}(\tau))}$ blocks, and \mathcal{S} is X -symmetric. Let $(\tau, \sigma) \in \text{Arc}(\Sigma)$. Then, since Σ is X -symmetric, $(\tau, \sigma)^y = (\sigma, \tau)$ for some $y \in X$. Set $\mathcal{S}_{(\tau, \sigma)} = \{\mathfrak{s}(\tau, S) \in \mathcal{S}_{\tau} \mid \sigma \in S\}$. Then $\frac{r}{m(\mathcal{D}(\tau))} = |\mathcal{S}_{(\tau, \sigma)}|$ is odd, $\mathcal{S}_{(\tau, \sigma)}^y = \mathcal{S}_{(\sigma, \tau)}$ and $\mathcal{S}_{(\tau, \sigma)}^{y^2} = \mathcal{S}_{(\tau, \sigma)}$. Let \mathcal{O} be a $\langle y^2 \rangle$ -orbit on $\mathcal{S}_{(\tau, \sigma)}$ with odd length l . Then for $\mathfrak{l} \in \mathcal{O}$, the stabilizer of \mathfrak{l} in $\langle y^2 \rangle$ is $\langle y^{2l} \rangle$. Let $z = y^l$ and $\mathfrak{r} = \mathfrak{l}^z$. Then $(\mathfrak{l}, \mathfrak{r})^z = (\mathfrak{r}, \mathfrak{l})$, and hence $\Theta := \{(\mathfrak{l}^x, \mathfrak{r}^x) \mid x \in X\}$ is a self-paired X -symmetric orbit on $DSt^k(\Sigma)$ with $St(\Theta) = \mathcal{S}$. \blacksquare

The following Theorem 3.7 says that, for any X -symmetric graph Γ with an nontrivial X -invariant partition, Γ or a quotient of Γ can be constructed as in Construction 3.4.

Let $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$. For $B \in \mathcal{B}$ and $\mathfrak{v} \in B$, define $B_{\mathfrak{v}} = B \cap (\cap_{C \in \Gamma_{\mathcal{B}}(\mathfrak{v})} \Gamma(C))$. Then $|B_{\mathfrak{v}}|$, denoted by $m^*(\Gamma, \mathcal{B})$ is independent of the choices of B and \mathfrak{v} . Noting that Γ is not a multicover of $\Gamma_{\mathcal{B}}$, we have $m^*(\Gamma, \mathcal{B}) \leq k := |B \cap \Gamma(C)|$ for $C \in \Gamma_{\mathcal{B}}(B)$. In fact, $m^*(\Gamma, \mathcal{B})$ is the multiplicity of the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$. Set $\underline{\mathcal{B}} = \{B_{\mathfrak{v}} \mid B \in \mathcal{B}, \mathfrak{v} \in B\}$. Then $\underline{\mathcal{B}}$ is an X -invariant

partition of $V(\Gamma)$. For $B \in \mathcal{B}$, we set $\bar{B} = \{B_v \mid v \in B\}$. Then $\Gamma_{\underline{\mathcal{B}}}$ is an X -symmetric graph with an X -invariant partition $\bar{\mathcal{B}} := \{\bar{B} \mid B \in \mathcal{B}\}$ such that $(\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}} \cong \Gamma_{\mathcal{B}}$. Moreover, $m^*(\Gamma_{\underline{\mathcal{B}}}, \bar{\mathcal{B}}) = 1$.

Theorem 3.7. *Let $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$. Set $\mathcal{S} = \{\mathfrak{s}(B, \Gamma_{\mathcal{B}}(v)) \mid B \in \mathcal{B}, v \in B\}$. Then \mathcal{S} is an X -symmetric orbit on $St^r(\Gamma_{\mathcal{B}})$, where $r = |\Gamma_{\mathcal{B}}(v)|$ is a constant. Let $\Theta = \{(\mathfrak{l}, \mathfrak{r}) \mid \mathfrak{l} = \mathfrak{s}(B, \Gamma_{\mathcal{B}}(v)), \mathfrak{r} = \mathfrak{s}(C, \Gamma_{\mathcal{B}}(u)), v \in B \in \mathcal{B}, u \in C \in \mathcal{B}, (v, u) \in \text{Arc}(\Gamma)\}$. Then Θ is a self-paired X -symmetric orbit on $DSt^r(\Gamma_{\mathcal{B}})$ with $St(\Theta) = \mathcal{S}$ and $\Gamma_{\underline{\mathcal{B}}} \cong \Pi(\Gamma_{\mathcal{B}}, \Theta)$, and X acts faithfully on \mathcal{B} if and only if X acts faithfully on $\underline{\mathcal{B}}$.*

Proof. It is easily shown that Θ is a self-paired X -symmetric orbit on $DSt^r(\Gamma_{\mathcal{B}})$ with $St(\Theta) = \mathcal{S}$. Assume $m^*(\Gamma, \mathcal{B}) = 1$. Then, for two distinct vertices $v \in B \in \mathcal{B}$ and $u \in C \in \mathcal{B}$ of Γ , $B_v = \{v\}$ and $C_u = \{u\}$, it implies $\Gamma_{\mathcal{B}}(v) \neq \Gamma_{\mathcal{B}}(u)$, and hence $\mathfrak{s}(B, \Gamma_{\mathcal{B}}(v)) \neq \mathfrak{s}(C, \Gamma_{\mathcal{B}}(u))$. Thus $V(\Gamma) \rightarrow V(\Pi(\Gamma_{\mathcal{B}})), v \mapsto \mathfrak{s}(B, \Gamma_{\mathcal{B}}(v))$ is a bijection. Further, it is easy to see this bijection is in fact an isomorphism between Γ and $\Pi(\Gamma_{\mathcal{B}}, \Theta)$.

Now assume $m^*(\Gamma, \mathcal{B}) > 1$. Recall that $m^*(\Gamma, \mathcal{B}) \leq k := |B \cap \Gamma(C)|$ for $C \in \Gamma_{\mathcal{B}}(B)$. Then $\underline{\mathcal{B}}$ is a proper refinement of \mathcal{B} . Consider $\Gamma_{\underline{\mathcal{B}}}$ with X -invariant partition $\bar{\mathcal{B}}$. Then $m^*(\Gamma_{\underline{\mathcal{B}}}, \bar{\mathcal{B}}) = 1$. Then a similar argument as above leads to $\Gamma_{\underline{\mathcal{B}}} \cong \Pi(\Sigma, \bar{\Theta})$, where $\Sigma = (\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}}$ and $\bar{\Theta} = \{(\bar{\mathfrak{l}}, \bar{\mathfrak{r}}) \mid \bar{\mathfrak{l}} = \mathfrak{s}(\bar{B}, \Sigma(B_v)), \bar{\mathfrak{r}} = \mathfrak{s}(\bar{C}, \Sigma(C_u)), B_v \in \bar{B} \in \bar{\mathcal{B}}, C_u \in \bar{C} \in \bar{\mathcal{B}}, (B_v, C_u) \in \text{Arc}(\Gamma_{\underline{\mathcal{B}}})\}$. Noting that $B_v = B_{v'}$ for any $v' \in B_v$, it follows that $\mathfrak{s}(\bar{B}, \Sigma(B_v)) \mapsto \mathfrak{s}(B, \Gamma_{\mathcal{B}}(v))$ gives a bijection between $V(\Pi(\Sigma, \bar{\Theta}))$ and $V(\Pi(\Gamma_{\mathcal{B}}, \Theta))$, which is in fact an isomorphism between $\Pi(\Sigma, \bar{\Theta})$ and $\Pi(\Gamma_{\mathcal{B}}, \Theta)$. Hence $\Gamma_{\underline{\mathcal{B}}} \cong \Pi(\Gamma_{\mathcal{B}}, \Theta)$.

Let K and H be the kernels of X acting on \mathcal{B} and on $\underline{\mathcal{B}}$ respectively. Noting that $\underline{\mathcal{B}}$ is a refinement of \mathcal{B} , we have $H \leq K$. Let $x \in K$ and $B_v \in \bar{B} \in \underline{\mathcal{B}}$. Since $m^*(\Gamma_{\underline{\mathcal{B}}}, \bar{\mathcal{B}}) = 1$, we have $\{B_v\} = \bar{B} \cap (\cap_{\bar{C} \in (\Gamma_{\underline{\mathcal{B}}})_{\bar{\mathcal{B}}}(B_v)} \Gamma_{\underline{\mathcal{B}}}(\bar{C})) = \bar{B} \cap (\cap_{C \in \Gamma_{\mathcal{B}}(v)} \Gamma_{\underline{\mathcal{B}}}(\bar{C}))$, yielding $B_v^x = B_v$. The above argument implies $x \in H$. Hence $K \leq H$, and so $H = K$. Therefore, X acts faithfully on \mathcal{B} (that is, $K = 1$) if and only if X acts faithfully on $\underline{\mathcal{B}}$ (that is, $H = 1$). \blacksquare

Finally, we list a simple fact which will be used in the following sections.

Theorem 3.8. *Let $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$ and $B \in \mathcal{B}$. If $m^*(\Gamma, \mathcal{B}) = 1$ and $m(\mathcal{D}(B)) = 1$, then $X_B^B \cong X_B^{\Gamma_{\mathcal{B}}(B)}$.*

Proof. If $x \in X$ fixes B set-wise, then it also fixes the neighborhood $\Gamma_{\mathcal{B}}(B)$ of B in $\Gamma_{\mathcal{B}}$. Now consider the action of X_B on $\Gamma_{\mathcal{B}}(B)$, and let K be the kernel of this action. For any $v \in B$, since $m^*(\Gamma, \mathcal{B}) = 1$, we have $\{v\} = B \cap (\cap_{C \in \Gamma_{\mathcal{B}}(v)} \Gamma(C))$. It follows that K fixes v . Thus $K \leq X_{(B)}$. On the other hand, x fixes $B \cap \Gamma(C)$ point-wise for any $x \in X_{(B)}$ and any $C \in \Gamma_{\mathcal{B}}(B)$, in particular, $B \cap \Gamma(C^x) = (B \cap \Gamma(C))^x = B \cap \Gamma(C)$. It follows from $m(\mathcal{D}(B)) = 1$ that $C = C^x$. Therefore, $x \in K$. Thus $X_{(B)} \leq K$, and so $X_{(B)} = K$. Then $X_B^B \cong X_B/X_{(B)} = X_B/K \cong X_B^{\Gamma_{\mathcal{B}}(B)}$. \blacksquare

4. THE MAIN RESULT

We state the main result of this paper in this section and prove it in the next four sections.

To state the result we need the following concept. A *near n -gonal graph* [15] is a connected graph Σ of girth at least 4 together with a set \mathcal{E} of n -cycles of Σ such that each 2-arc of Σ is contained in a unique member of \mathcal{E} .

Let $(\Gamma, X, \mathcal{B}) \in \mathcal{G}$. For a subgraph Δ of Γ , denote by $X_{[\Delta]}$ the subgroup of X which preserves the adjacency of Δ , and set $X_{[\Delta]}^{[\Delta]} = X_{[\Delta]}/X_{(V(\Delta))}$. Recall that, for $\mathfrak{v} \in B \in \mathcal{B}$ and $C \in \Gamma_{\mathcal{B}}(B)$, the parameters $v := |B|$, $k := |\Gamma(C) \cap B|$, $r := |\Gamma_{\mathcal{B}}(\mathfrak{v})|$ and $b := \text{val}(\Gamma_{\mathcal{B}})$ are independent of the choices of B and \mathfrak{v} , and $\mathcal{D}(B)$ is an X_B -flag-transitive 1 -(v, k, r) design with b blocks. Now we are ready to state the main result of this paper.

Theorem 4.1. *Let $(\Gamma, X, \mathcal{B}) \in \hat{\mathcal{G}}$ and $B \in \mathcal{B}$. Let $e = |E(\Gamma_{\mathcal{B}})|$, $\mu = |V(\Gamma_{\mathcal{B}})|$. Suppose that $k = 3$. Then $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive if and only if one of the following four cases occurs.*

- (a) $(v, b, r) = (4, 4, 3)$, $X_B^B \cong A_4$ or S_4 ;
- (b) $(v, b, r) = (6, 4, 2)$, $X_B^B \cong A_4$ or S_4 ;
- (c) $(v, b, r) = (7, 7, 3)$, $X_B^B \cong \text{PSL}(3, 2)$;
- (d) $v = 3b \geq 6$, $r = 1$ and X_B acts 2-transitively on the blocks of $\mathcal{D}(B)$.

Furthermore, if case (a) occurs, then $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired X -orbit Δ on $\text{Arc}_3(\Gamma_{\mathcal{B}})$, X acts faithfully on \mathcal{B} , and any connected tetravalent $(X, 2)$ -arc-transitive graph can occur as $\Gamma_{\mathcal{B}}$; moreover, one of the following three statements holds.

- (a.1) $\Gamma[B, C] \cong 3K_2$, $\text{val}(\Gamma) = 3$, there exists an X -orbit \mathcal{E} of n -cycles of $\Gamma_{\mathcal{B}}$ with $|\mathcal{E}| = m$ such that $\Delta = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Gamma_{\mathcal{B}})$ with $mn = 3e = 6\mu$. Moreover, either $\Gamma_{\mathcal{B}} \cong K_5$ or $\Gamma_{\mathcal{B}}$ is a near n -gonal graph with respect to \mathcal{E} ; either $X_B \cong A_4$, Γ is $(X, 1)$ -arc-regular and $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-regular; or $X_B \cong S_4$ and Γ is $(X, 2)$ -arc-regular.
- (a.2) $\Gamma[B, C] \cong K_{3,3} - 3K_2$, $\text{val}(\Gamma) = 6$, $X_B \cong S_4$, and Γ is connected and $(X, 1)$ -arc-regular. Further, $\Delta' := \text{Arc}_3(\Gamma_{\mathcal{B}}) \setminus \Delta$ is a self-paired X -orbit on $\text{Arc}_3(\Gamma_{\mathcal{B}})$, and there exists an X -orbit \mathcal{E} of n -cycles of Σ with $|\mathcal{E}| = m$, such that $\Delta' = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Sigma)$ with $mn = 3e = 6\mu$. Moreover, either $\Gamma_{\mathcal{B}} \cong K_5$ or $\Gamma_{\mathcal{B}}$ is a near n -gonal graph.
- (a.3) $\Gamma[B, C] \cong K_{3,3}$, $\text{val}(\Gamma) = 9$, Γ is connected and $(X, 1)$ -transitive, and $\Gamma_{\mathcal{B}}$ is $(X, 3)$ -arc transitive.

If case (b) holds, then $\Gamma \cong \mathbb{J}(\Gamma_{\mathcal{B}}, \Delta) \cong \Psi(\Gamma_{\mathcal{B}}, \Lambda)$ for some self-paired X -orbit Δ on $\text{Arc}_3(\Gamma_{\mathcal{B}})$ and some self-paired X -orbit Λ on $J(\Gamma_{\mathcal{B}})$, X acts faithfully

on \mathcal{B} , and any connected tetravalent $(X, 2)$ -arc-transitive graph can occur as $\Gamma_{\mathcal{B}}$; moreover, one of the following three cases occurs.

- (b.1) $\Gamma[B, C] \cong 3K_2$, $\Gamma \cong mC_n$, and there exists an X -orbit \mathcal{E} of n -cycles of $\Gamma_{\mathcal{B}}$ with $|\mathcal{E}| = m$, such that $\Delta = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Gamma_{\mathcal{B}})$ with $mn = 3e = 6\mu$. Moreover, either $\Gamma_{\mathcal{B}} \cong K_5$ or $\Gamma_{\mathcal{B}}$ is a near n -gonal graph with respect to \mathcal{E} ; either $X_B \cong A_4$, Γ is $(X, 1)$ -arc-regular and $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-regular, or $X_B \cong S_4$ and Γ is not $(X, 1)$ -arc-regular.
- (b.2) $\Gamma[B, C] \cong K_{3,3} - 3K_2$, $\text{val}(\Gamma) = 4$, $X_B \cong S_4$, Γ is connected and $(X, 1)$ -arc-regular. Further, $\Delta' := \text{Arc}_3(\Gamma_{\mathcal{B}}) \setminus \Delta$ is a self-paired X -orbit on $\text{Arc}_3(\Gamma_{\mathcal{B}})$, and there exists an X -orbit \mathcal{E} of n -cycles of Σ with $|\mathcal{E}| = m$, such that $\Delta' = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Sigma)$ with $mn = 3e = 6\mu$. Moreover, either $\Gamma_{\mathcal{B}} \cong K_5$ or $\Gamma_{\mathcal{B}}$ is a near n -gonal graph.
- (b.3) $\Gamma[B, C] \cong K_{3,3}$, $\text{val}(\Gamma) = 6$, Γ is connected and $(X, 1)$ -transitive, and $\Gamma_{\mathcal{B}}$ is $(X, 3)$ -arc transitive.

If case (c) holds, then $\Gamma \cong \Pi(\Gamma_{\mathcal{B}}, \Theta)$ for some self-paired X -symmetric orbit Θ on $DSt^3(\Gamma_{\mathcal{B}})$, X acts faithfully on \mathcal{B} , one connected heptavalent X -symmetric graph Σ can occur as $\Gamma_{\mathcal{B}}$, if and only if $X_{\tau}^{\Sigma(\tau)} \cong PSL(3, 2)$ for $\tau \in V(\Sigma)$; further, one of the following three cases occurs.

- (c.1) $\Gamma[B, C] \cong 3K_2$, $\text{val}(\Gamma) = 3$, Γ is $(X, 2)$ -arc-transitive but not $(X, 2)$ -arc-regular.
- (c.2) $\Gamma[B, C] \cong K_{3,3} - 3K_2$, $\text{val}(\Gamma) = 6$, Γ is connected and is $(X, 1)$ -transitive.
- (c.3) $\Gamma[B, C] \cong K_{3,3}$, $\text{val}(\Gamma) = 9$, Γ is connected and is $(X, 1)$ -transitive.

If case (d) occurs, then one of the following three cases occurs.

- (d.1) $\Gamma[B, C] \cong 3K_2$, $\Gamma \cong 3eK_2$.
- (d.2) $\Gamma[B, C] \cong K_{3,3} - 3K_2$, $\Gamma \cong e(K_{3,3} - 3K_2)$.
- (d.3) $\Gamma[B, C] \cong K_{3,3}$, $\Gamma \cong eK_{3,3}$.

5. SELF-PAIRED ORBITS OF 3-ARCS

We begin this section by showing that there always exists a self-paired X -orbit of 3-arcs for any symmetric graph of even valency.

Theorem 5.1. *Any X -symmetric graph Σ of even valency $v \geq 2$ contains a self-paired X -orbit Δ on $\text{Arc}_3(\Sigma)$.*

Proof. For any $(\tau, \sigma) \in \text{Arc}(\Sigma)$, as Σ is X -symmetric, there exists $y \in X$ such that $(\tau, \sigma)^y = (\sigma, \tau)$, and so $(\Sigma(\tau) \setminus \{\sigma\})^y = \Sigma(\sigma) \setminus \{\tau\}$, $(\Sigma(\tau) \setminus \{\sigma\})^{y^2} = \Sigma(\tau) \setminus \{\sigma\}$. Since $|\Sigma(\tau) \setminus \{\sigma\}| = v - 1$ is odd, there must be some $\langle y^2 \rangle$ -orbit \mathcal{O} on $\Sigma(\tau) \setminus \{\sigma\}$ with odd length l . For $\tau_1 \in \mathcal{O}$, the stabilizer of

τ_1 in $\langle y^2 \rangle$ is $\langle y^{2l} \rangle$. Let $z = y^l$, $\sigma_1 = \tau_1^z$ and $\mathfrak{x} = (\tau_1, \tau, \sigma, \sigma_1)$. Since l is odd, $(\tau, \sigma)^z = (\tau, \sigma)^{y^l} = (\sigma, \tau)$. Then $\mathfrak{x} \in \text{Arc}_3(\Sigma)$ and $\mathfrak{x}^z = \mathfrak{x}^{-1}$. Thus $\Delta = \{(\tau_1^x, \tau^x, \sigma^x, \sigma_1^x) \mid x \in X\}$ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$. \blacksquare

Let Σ be an X -symmetric graph with valency $\mathfrak{v} \geq 2$ and Δ be an X -orbit on $\text{Arc}_3(\Sigma)$. For any $\mathfrak{x} := (\tau_1, \tau, \sigma, \sigma_1) \in \Delta$, consider the action of $X_{(\tau_1, \tau, \sigma)}$ on $\Sigma(\sigma) \setminus \{\tau\}$, and denote by O_1, O_2, \dots, O_t the orbits of this action. Without loss of generality, assume $\sigma_1 \in O_1$ and $|O_2| \leq |O_3| \leq \dots \leq |O_t|$. Since Δ is an X -orbit of 3-arcs, all $\ell_i(\Delta) := |O_i| \geq 1$ are independent of the choice of $\mathfrak{x} \in \Delta$. Set $\mathbf{l}(\Delta) = (\ell_1(\Delta), \dots, \ell_t(\Delta))$.

Theorem 5.2. *Let Σ be a connected $(X, 2)$ -arc-transitive graph with valency $\mathfrak{v} \geq 3$ and Δ be a self-paired X -orbit on $\text{Arc}_3(\Sigma)$ such that $\ell_1(\Delta) = 1$. If X is faithful on $V(\Sigma)$, then X_τ is faithful on $\Sigma(\tau)$ for $\tau \in V(\Sigma)$. Set $\mu = |V(\Sigma)|$ and $e = |E(\Sigma)|$. Then $\mathfrak{J}(\Sigma, \Delta) \cong mC_n$ such that*

- (1) $m \geq \mathfrak{v}(\mathfrak{v}-1)/2$, $n \geq \text{girth}(\Sigma) \geq 3$ and $mn = \mu\mathfrak{v}(\mathfrak{v}-1)/2 = e(\mathfrak{v}-1)$;
- (2) there exists an X -orbit \mathcal{E} of n -cycles of Σ with $\Delta = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$ and $|\mathcal{E}| = m$;
- (3) $X_{[C]}^{[C]} \cong D_{2n}$ for $C \in \mathcal{E}$, where D_{2n} is the dihedral group of order $2n$;
- (4) every 2-path of Σ is contained in a unique member of \mathcal{E} , and either $\Sigma \cong K_{\mathfrak{v}+1}$ (the complete graph on $\mathfrak{v}+1$ vertices), or $n \geq \text{girth}(\Sigma) \geq 4$ and Σ is a near n -gonal graph with respect to \mathcal{E} .

Proof. Since Σ is $(X, 2)$ -arc-transitive, every 2-arc of Σ lies in a member of Δ . Let (τ, σ) be an arbitrary arc of Σ . Since $\ell_1(\Delta) = 1$ and Δ is a self-paired X -orbit, we conclude that, for any $\tau_1 \in \Sigma(\tau) \setminus \{\sigma\}$ there is a unique $\sigma_1 \in \Sigma(\sigma) \setminus \{\tau\}$ such that $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$, $X_{(\tau_1, \tau, \sigma)} = X_{(\tau, \sigma, \sigma_1)}$, and that $(\tau'_1, \tau, \sigma, \sigma_1) \in \Delta$ implies $\tau'_1 = \tau$. Then $(X_\tau)_{(\Sigma(\tau))} = \cap_{\tau_1 \in \Sigma(\tau) \setminus \{\sigma\}} X_{(\tau_1, \tau, \sigma)} = \cap_{\sigma_1 \in \Sigma(\sigma) \setminus \{\tau\}} X_{(\tau, \sigma, \sigma_1)} = (X_\sigma)_{(\Sigma(\sigma))}$. It follows from the connectedness of Σ that $(X_\tau)_{(\Sigma(\tau))}$ fixes every vertex of Σ . Thus, if X is faithful on $V(\Sigma)$, then $(X_\tau)_{(\Sigma(\tau))} = 1$ and X_τ is faithful on $\Sigma(\tau)$.

Let $\mathfrak{J} = \mathfrak{J}(\Sigma, \Delta)$. By Proposition 2.1, \mathfrak{J} is X -symmetric and admits an X -invariant partition $\mathcal{P} := \{P_\sigma \mid \sigma \in V(\Sigma)\}$ such that $\Sigma \cong \mathfrak{J}_\mathcal{P}$, where P_σ is the set of 2-paths of Σ with middle vertex σ . It follows from [14] that $r := |\mathfrak{J}_\mathcal{P}(\mathfrak{v})| = 2$ and $\lambda := |P_\delta \cap \mathfrak{J}(P_\tau) \cap \mathfrak{J}(P_\sigma)| = 1$ for any vertex \mathfrak{v} (a 2-path of Σ) in $V(\mathfrak{J})$ and P_δ with $\mathfrak{v} \in P_\delta$ and $\mathfrak{J}_\mathcal{P}(\mathfrak{v}) = \{P_\tau, P_\sigma\}$. Since $\ell_1(\Delta) = 1$ and Δ is self-paired, for any 2-path $[\tau_1, \tau, \sigma]$ of Σ , there exist exactly two 2-paths $[\tau, \sigma, \sigma_1]$ and $[\tau_2, \tau_1, \tau]$ such that $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$ and $(\tau_2, \tau_1, \tau, \sigma) \in \Delta$. It follows that \mathfrak{J} is of valency two, and that \mathfrak{J} is a disjoint union of cycles. Assume $\mathfrak{J} \cong mC_n$. Then mn is the number of 2-paths of Σ , and hence $mn = \mu\mathfrak{v}(\mathfrak{v}-1)/2 = e(\mathfrak{v}-1)$. Noting that \mathfrak{J} is of valency 2 and every P_σ is an independent set of $V(\mathfrak{J})$, it follows that different vertices in P_σ appear in different n -cycles of \mathfrak{J} . Thus $m \geq |P_\sigma| = \mathfrak{v}(\mathfrak{v}-1)/2$.

Let $\mathfrak{C} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}_1]$ be an arbitrary n -cycle of \mathfrak{J} , where $\mathbf{v}_i = [\tau_i, \sigma_i, \delta_i]$ are n distinct 2-paths of Γ with middle vertices σ_i , respectively. Without loss of generality, we assume $\delta_i = \sigma_{i+1} = \tau_{i+2}$ for $1 \leq i \leq n$, where subscripts are reduced modulo n . Since \mathbf{v}_i is a 2-path, $\sigma_i \neq \delta_i$, hence $\sigma_i \neq \sigma_{i+1}$. Then $(\sigma_i, \sigma_{i+1}) \in \text{Arc}(\Sigma)$. Since $\{\mathbf{v}_i, \mathbf{v}_{i+1}\}$ is an edge of \mathfrak{J} , we have $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2}) = (\tau_i, \sigma_i, \delta_i, \delta_{i+1}) \in \Delta$.

Now we shall show $C = [\sigma_1, \sigma_2, \dots, \sigma_n, \sigma_1]$ is an n -cycle of Σ . In particular, $n \geq \text{girth}(\Sigma) \geq 3$. Note that \mathfrak{C} is a component of \mathfrak{J} . Then \mathfrak{C} is $X_{[\mathfrak{C}]}$ -symmetric; in particular, $X_{[\mathfrak{C}]}^{[\mathfrak{C}]} \cong D_{2n}$, the dihedral group of order $2n$. Thus there exist $x, y \in X_{[\mathfrak{C}]}$ such that $\mathbf{v}_i^x = \mathbf{v}_{i+1}$ and $\mathbf{v}_i^y = \mathbf{v}_{n-i+1}$, hence $\sigma_i^x = \sigma_{i+1}$ and $\sigma_i^y = \sigma_{n-i+1}$ for $1 \leq i \leq n$ with subscripts modulo n . Assume that $\sigma_i = \sigma_j$ for some i and j . Then $\sigma_{i+1} = \sigma_i^x = \sigma_j^x = \sigma_{j+1}$ and $\sigma_{i+2} = \sigma_{i+1}^x = \sigma_{j+1}^x = \sigma_{j+2}$. Thus $P_{\sigma_i} = P_{\sigma_j}$, $P_{\sigma_{i+1}} = P_{\sigma_{j+1}}$ and $P_{\sigma_{i+2}} = P_{\sigma_{j+2}}$. It yields $(\mathbf{v}_i, \mathbf{v}_{i+1}), (\mathbf{v}_j, \mathbf{v}_{j+1}) \in \text{Arc}(\mathfrak{J}[P_{\sigma_i}, P_{\sigma_{i+1}}])$ and $(\mathbf{v}_{i+1}, \mathbf{v}_{i+2}), (\mathbf{v}_{j+1}, \mathbf{v}_{j+2}) \in \text{Arc}(\mathfrak{J}[P_{\sigma_{i+1}}, P_{\sigma_{i+2}}])$. It follows that $\mathbf{v}_{i+1}, \mathbf{v}_{j+1} \in P_{\sigma_{i+1}} \cap \mathfrak{J}(P_{\sigma_i}) \cap \mathfrak{J}(P_{\sigma_{i+2}})$. Since $1 = \lambda = |P_{\sigma_{i+1}} \cap \mathfrak{J}(P_{\sigma_i}) \cap \mathfrak{J}(P_{\sigma_{i+2}})|$, we have $\mathbf{v}_{i+1} = \mathbf{v}_{j+1}$. Thus $i = j$. Then all σ_i are distinct, C is an n -cycle and C is $\langle x, y \rangle$ -symmetric. It implies $X_{[C]}^{[C]} \cong D_{2n}$. Hence $X_{[C^g]}^{[C^g]} \cong D_{2n}$ for any $g \in X$.

Set $\mathcal{E} = \{C^x \mid x \in X\}$. Then \mathcal{E} is an X -orbit of n -cycles of Σ . Since C is $X_{[C]}$ -symmetric, C is $(X_{[C]}, 3)$ -arc-transitive. Recall that the 3-arc $(\sigma_{i-1}, \sigma_i, \sigma_{i+1}, \sigma_{i+2})$ of C is contained in Δ . It follows that $\Delta = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$.

It is easily shown that $X_{[\mathfrak{C}]}$ is a subgroup of $X_{[C]}$, and so $|\mathcal{E}| = |X : X_{[C]}| \leq |X : X_{[\mathfrak{C}]}| = m$. Suppose that $X_{[\mathfrak{C}]}$ is a proper subgroup of $X_{[C]}$. Then there is some $z \in X_{[C]}$ with $C^z = C$ but $\mathfrak{C}^z \neq \mathfrak{C}$. Noting that \mathfrak{C} and \mathfrak{C}^z are distinct connected component of \mathfrak{J} , we have $V(\mathfrak{C}) \cap V(\mathfrak{C}^z) = \emptyset$. Since $C^z = C$, there exist i, j and l with $\sigma_1 = \sigma_i^z, \sigma_2 = \sigma_j^z$ and $\sigma_3 = \sigma_l^z$. Then $\mathbf{v}_i^z = [\tau_i^z, \sigma_1, \delta_i^z] \in P_{\sigma_1}$, $\mathbf{v}_j^z = [\tau_j^z, \sigma_2, \delta_j^z] \in P_{\sigma_2}$ and $\mathbf{v}_l^z = [\tau_l^z, \sigma_3, \delta_l^z] \in P_{\sigma_3}$. Since $(\sigma_1, \sigma_2, \sigma_3)$ is a 2-arc of C , we know $(\sigma_i, \sigma_j, \sigma_l)$ is also a 2-arc of C . It follows that $i - j \equiv j - l \equiv \pm 1 \pmod{n}$. Then $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_l]$ is a 2-path of \mathfrak{C} , and so $[\mathbf{v}_i^z, \mathbf{v}_j^z, \mathbf{v}_l^z]$ is a 2-path of \mathfrak{C}^z . Thus $\mathbf{v}_2, \mathbf{v}_j^z \in P_{\sigma_2} \cap \mathfrak{J}(P_{\sigma_1}) \cap \mathfrak{J}(P_{\sigma_3})$. Since $V(\mathfrak{C}) \cap V(\mathfrak{C}^z) = \emptyset$, we have $\mathbf{v}_2 \neq \mathbf{v}_j^z$, which contradicts $\lambda = 1$. $X_{[\mathfrak{C}]} = X_{[C]}$ and so $|\mathcal{E}| = |X : X_{[C]}| = |X : X_{[\mathfrak{C}]}| = m$.

Since Σ is $(X, 2)$ -arc-transitive, every 2-path is contained in some n -cycle in \mathcal{E} . Then $mn = |\text{Path}_2(\Sigma)| = |\cup_{C \in \mathcal{E}} \text{Path}_2(C)| \leq \sum_{C \in \mathcal{E}} |\text{Path}_2(C)| = mn$. It follows that every 2-path of Σ is contained in a unique member of \mathcal{E} . Thus either $\text{girth}(\Sigma) = 3$ and $\Sigma \cong K_{v+1}$, or $n \geq \text{girth}(\Sigma) \geq 4$ and Σ is a near n -gonal graph with respect to \mathcal{E} . \blacksquare

The following result follows from Theorem 5.1 and 5.2.

Corollary 5.3. *Any connected $(X, 2)$ -arc-regular graph with even valency and girth no less than 4 is a near n -gonal graph for some integer $n \geq 4$.*

6. ON TETRAVALENT SYMMETRIC GRAPHS

Let Σ be a regular graph with valency four. Recall that $J(\Sigma)$ is the set of pairs $([\tau', \tau, \tau''], [\sigma', \sigma, \sigma''])$ of 2-paths of Σ such that $\sigma \in \Sigma(\tau) \setminus \{\tau', \tau''\}$, $\tau \in \Sigma(\sigma) \setminus \{\sigma', \sigma''\}$. For an arbitrary 3-arc $\alpha := (\tau_1, \tau, \sigma, \sigma_1)$ of (Σ) , let J_α be the pair $([\tau_2, \tau, \tau_3], [\sigma_2, \sigma, \sigma_3])$ of 2-paths of Σ , where $\Sigma(\tau) = \{\sigma, \tau_1, \tau_2, \tau_3\}$ and $\Sigma(\sigma) = \{\tau, \sigma_1, \sigma_2, \sigma_3\}$. Then $J_\alpha \in J(\Sigma)$. For any subset Δ of $\text{Arc}_3(\Sigma)$, we set $J(\Delta) := \{J_\alpha \mid \alpha \in \Delta\}$. It is easily shown that Δ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$ if and only if $J(\Delta)$ is a self-paired X -orbit on $J(\Sigma)$.

Theorem 6.1. *Let Σ be a connected $(X, 2)$ -arc-transitive graph of valency 4. If Δ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$, then $\mathfrak{J}(\Sigma, \Delta) \cong \Psi(\Sigma, J(\Delta))$.*

Proof. Define $\psi : \text{Path}_2(\Sigma) \rightarrow \text{Path}_2(\Sigma); [\tau_1, \tau, \tau_2] \mapsto [\tau_3, \tau, \tau_4]$, where $\{\tau_3, \tau_4\} = \Sigma(\tau) \setminus \{\tau_1, \tau_2\}$. It is easy to check that ψ is an isomorphism from $\mathfrak{J}(\Sigma, \Delta)$ to $\Psi(\Sigma, J(\Delta))$. \blacksquare

The main aim of this section is to give a characterization of tetravalent $(X, 2)$ -arc-transitive graphs. The following simple lemma is useful.

Lemma 6.2. *Let Γ be an X -symmetric graph with an X -invariant partition \mathcal{B} such that $\Gamma_{\mathcal{B}}$ is connected and $(X, 2)$ -arc-transitive. Let $B \in \mathcal{B}$ and $C, D \in \Gamma_{\mathcal{B}}(B)$ with $C \neq D$. If $\Gamma[B, C]$ is connected and $\Gamma(C) \cap B \cap \Gamma(D) \neq \emptyset$, then Γ must be connected.*

Proof. It suffices to show that there is a path in Γ between any two different vertices \mathbf{v} and \mathbf{u} of Γ . Since $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive, $\Gamma[B, C]$ is independent of the choices of B and $C \in \Gamma_{\mathcal{B}}(B)$ up to isomorphism; and $|\Gamma(C) \cap B \cap \Gamma(D)|$ is independent of the choices of B and $C, D \in \Gamma_{\mathcal{B}}(B)$ (with $C \neq D$).

Assume first $\mathbf{v}, \mathbf{u} \in B$. Without loss of generality, we assume $\mathbf{v} \in \Gamma(C) \cap B \cap \Gamma(D)$. If $\mathbf{u} \in \Gamma(C) \cap B$, then there is a path in Γ between \mathbf{v} and \mathbf{u} as $\Gamma[B, C]$ is connected. So we assume $\mathbf{u} \notin \Gamma(C) \cap B$. Take $E \in \Gamma_{\mathcal{B}}(\mathbf{u})$. Then $E \in \Gamma_{\mathcal{B}}(B)$, $\mathbf{u} \in B \cap \Gamma(E)$ and $|\Gamma(C) \cap B \cap \Gamma(E)| = |\Gamma(C) \cap B \cap \Gamma(D)| > 0$. Let $\mathbf{w} \in \Gamma(C) \cap B \cap \Gamma(E)$. Then either $\mathbf{v} = \mathbf{w}$ or there is a path between \mathbf{v} and \mathbf{w} , and there is a path between \mathbf{w} and \mathbf{u} . Thus there is a path between \mathbf{v} and \mathbf{u} .

Now let $\mathbf{v} \in B$ and $\mathbf{u} \in B'$ with $B \neq B'$. Since $\Gamma_{\mathcal{B}}$ is connected, there is a path $[B = B_1, \dots, B_l = B']$. Let $\mathbf{u}'_l \in B_l$ and $\mathbf{u}_{l-1} \in B_{l-1}$ such that $\{\mathbf{u}_{l-1}, \mathbf{u}'_l\} \in E(\Gamma)$. Thus there is a path between \mathbf{u}_{l-1} and \mathbf{u} . Then induction on l implies that there is a path between \mathbf{v} and \mathbf{u} . \blacksquare

Now we are ready to state and prove the main result of this section.

Theorem 6.3. *Let Σ be a connected $(X, 2)$ -arc-transitive graph with valency $v = 4$, where X acts faithfully on $V(\Sigma)$. Then Σ has a self-paired X -orbit Δ on $\text{Arc}_3(\Sigma)$. Set $\mathfrak{J} := \mathfrak{J}(\Sigma, \Delta)$, $\Xi := \Xi(\Sigma, \Delta)$, $e := |E(\Sigma)|$, $\mu := |V(\Sigma)|$. Let $(\tau, \sigma) \in \text{Arc}(\Sigma)$. Then one of the following cases occurs.*

- (a) $\mathfrak{J}[P_\tau, P_\sigma] \cong \Xi[A_\tau, A_\sigma] \cong 3K_2$, $\mathfrak{J}(\Sigma, \Delta) \cong mC_n$, $\text{val}(\Xi) = 3$, and there exists an X -orbit \mathcal{E} of n -cycles of Σ with $|\mathcal{E}| = m$, such that $\Delta = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Sigma)$ with $mn = 3e = 6\mu$. Moreover, either $\Sigma \cong K_5$ or Σ is a near n -gonal graph with respect to \mathcal{E} ; and, either
- (a.1) $X_{P_\tau} = X_{A_\tau} = X_\tau \cong A_4$, both \mathfrak{J} and Ξ are $(X, 1)$ -arc-regular and Σ is $(X, 2)$ -arc-regular; or
- (a.2) $X_\tau = X_{P_\tau} = X_{A_\tau} \cong S_4$, \mathfrak{J} is not $(X, 1)$ -arc-regular, Ξ is $(X, 2)$ -arc-regular.
- (b) $\mathfrak{J}[P_\tau, P_\sigma] \cong \Xi[A_\tau, A_\sigma] \cong K_{3,3} - 3K_2$, $\text{val}(\mathfrak{J}) = 4$, $\text{val}(\Xi) = 6$, $X_{P_\tau} = X_{A_\tau} = X_\tau \cong S_4$, both \mathfrak{J} and Ξ are connected and $(X, 1)$ -arc-regular. Further, $\Delta' := \text{Arc}_3(\Sigma) \setminus \Delta$ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$, and there exists an X -orbit \mathcal{E} of n -cycles of Σ with $|\mathcal{E}| = m$, such that $\Delta' = \cup_{C \in \mathcal{E}} \text{Arc}_3(C)$, $X_{[C]}^{[C]} \cong D_{2n}$ for each $C \in \mathcal{E}$, where $m \geq 6$ and $n \geq \text{girth}(\Sigma) \geq 3$ with $mn = 3e = 6\mu$. Moreover, either $\Sigma \cong K_5$ or Σ is a near n -gonal graph with respect to \mathcal{E} .
- (c) $\mathfrak{J}[P_\tau, P_\sigma] \cong \Xi[A_\tau, A_\sigma] \cong K_{3,3}$, $\text{val}(\mathfrak{J}) = 6$, $\text{val}(\Xi) = 9$, both \mathfrak{J} and Ξ are connected and $(X, 1)$ -transitive, and Σ is $(X, 3)$ -arc-transitive.

Proof. By Theorem 5.1, Σ has a self-paired X -orbit Δ on $\text{Arc}_3(\Sigma)$. Then, by Proposition 2.1, $\mathfrak{J} := \mathfrak{J}(\Sigma, \Delta)$ is X -symmetric and admits an X -invariant partition $\mathcal{P} := \{P_\sigma \mid \sigma \in V(\Sigma)\}$ with $\Sigma \cong \mathfrak{J}_{\mathcal{P}}$, and by Proposition 2.4, $\Xi := \Xi(\Sigma, \Delta)$ is X -symmetric and admits an X -invariant partition $\mathcal{A} := \{A_\sigma \mid \sigma \in V(\Sigma)\}$ with $\Sigma \cong \Xi_{\mathcal{A}}$. Let $\ell_i := \ell_i(\Delta)$, $i = 1, 2, \dots, t$, be defined as in Section 5. Then $t \leq 3$ as $\text{val}(\Sigma) = 4$.

Let $(\tau, \sigma) \in \text{Arc}(\Sigma)$. Then there is a 3-arc $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$ as Σ is X -symmetric. It follows that $\{[\tau_1, \tau, \sigma], [\tau, \sigma, \sigma_1]\}$ is an edge of $\mathfrak{J}[P_\tau, P_\sigma]$, and that $\{(\tau, \tau_1), (\sigma, \sigma_1)\}$ is an edge of $\Xi[A_\tau, A_\sigma]$. It is easily shown that $X_{(\tau, \sigma)} = X_\tau \cap X_\sigma = X_{P_\tau} \cap X_{P_\sigma}$ acts transitively on the edges of $\mathfrak{J}[P_\tau, P_\sigma]$. It implies that the stabilizer $(X_{(\tau, \sigma)})_{[\tau_1, \tau, \sigma]} = X_{(\tau_1, \tau, \sigma)}$ acts transitively on the neighborhood of $[\tau_1, \tau, \sigma]$ in $\mathfrak{J}[P_\tau, P_\sigma]$. Then the valency of $\mathfrak{J}[P_\tau, P_\sigma]$ equals to $|X_{(\tau_1, \tau, \sigma)} : (X_{(\tau_1, \tau, \sigma)})_{[\tau, \sigma, \sigma_1]}| = |X_{(\tau_1, \tau, \sigma)} : X_{(\tau_1, \tau, \sigma, \sigma_1)}| = \ell_1$. Further, since Σ is $(X, 2)$ -arc-transitive, $X_{(\tau, \sigma)}$ is transitive on $\Sigma(\tau) \setminus \{\sigma\} := \{\tau_1, \tau_2, \tau_3\}$ and on $\Sigma(\sigma) \setminus \{\tau\} := \{\sigma_1, \sigma_2, \sigma_3\}$. Thus $V(\mathfrak{J}[P_\tau, P_\sigma]) = \{[\tau_i, \tau, \sigma] \mid i = 1, 2, 3\} \cup \{[\tau, \sigma, \sigma_i] \mid i = 1, 2, 3\}$. A similar argument leads to $V(\Xi[A_\tau, A_\sigma]) = \{(\tau, \tau_i) \mid i = 1, 2, 3\} \cup \{(\sigma, \sigma_i) \mid i = 1, 2, 3\}$. It is easy to check that $[\tau_i, \tau, \sigma] \mapsto (\tau, \tau_i)$, $[\tau, \sigma, \sigma_i] \mapsto (\sigma, \sigma_i)$ gives an isomorphism from $\mathfrak{J}[P_\tau, P_\sigma]$ to $\Xi[A_\tau, A_\sigma]$. Further, $\mathfrak{J}[P_\tau, P_\sigma] \cong 3K_2$, $K_3 - 3K_2$ or $K_{3,3}$ according to $\ell_1 = 1, 2$ or 3 , respectively. By [14, Theorem 4.3], $2 = r_{\mathcal{P}} := |\mathfrak{J}_{\mathcal{P}}([\tau_1, \tau, \sigma])|$ for any $[\tau_1, \tau, \sigma] \in V(\mathfrak{J})$. Then $\text{val}(\mathfrak{J}) = r_{\mathcal{P}}\ell_1 = 2\ell_1$. By Lemma 2.5, $\text{val}(\Xi) = r_{\mathcal{A}}\ell_1 = 3\ell_1$. Since Σ is $(X, 2)$ -arc-transitive, $X_\tau^{\Sigma(\tau)} \cong A_4$ or S_4 . It is easy to see $X_\tau = X_{P_\tau} = X_{A_\tau}$, $(X_\tau)_{(\Sigma(\tau))} = X_{(P_\tau)} = X_{(A_\tau)}$ and hence $X_\tau^{\Sigma(\tau)} \cong X_{P_\tau}^{P_\tau} = X_{A_\tau}^{A_\tau}$. We treat the following three separate cases.

Case 1. $\ell_1 = 1$. Then $\text{val}(\mathfrak{J}) = 2$, $\text{val}(\Xi) = 3$ and $\mathbf{l}(\Delta) = (1, 1, 1)$ or $(1, 2)$ in this case. By Theorem 5.2, the part of (a) prior to (a.1) holds. Again by Theorem 5.2, X_τ acts faithfully on $\Sigma(\tau)$, and hence $X_\tau^{\Sigma(\tau)} \cong X_\tau$.

Assume first $\mathbf{l}(\Delta) = (1, 1, 1)$. Then $X_{(\tau_1, \tau, \sigma)} \leq X_{(\tau, \sigma, \sigma_1)}$ for any 2-arc (τ_1, τ, σ) of Σ and $\tau \neq \sigma_1 \in \Sigma(\sigma)$. Since Σ is $(X, 2)$ -arc-transitive, the stabilizers of any two 2-arcs of Σ are conjugate in X , in particular, they have the same order. Thus $X_{(\tau_1, \tau, \sigma)} = X_{(\tau, \sigma, \sigma_1)}$. Since Σ is connected, $X_{(\tau_1, \tau, \sigma)} = X_{(\tau'_1, \tau', \sigma')}$ for an arbitrary 2-arc $(\tau'_1, \tau', \sigma')$ of Σ . Hence $X_{(\tau_1, \tau, \sigma)} = 1$ as X is faithful on $V(\Sigma)$. Then Σ is $(X, 2)$ -arc-regular. It implies $X_{P_\tau} = X_{A_\tau} = X_\tau \cong A_4$. Then (a.1) follows from calculating the numbers of arcs or 2-arcs of \mathfrak{J} , Ξ and Σ .

Now let $\mathbf{l}(\Delta) = (1, 2)$. Then $X_{(\tau_1, \tau, \sigma)}$ acts transitively on $\Sigma(\sigma) \setminus \{\tau, \sigma_1\}$ for $(\tau_1, \tau, \sigma, \sigma_1) \in \Delta$. Thus $X_{(\tau_1, \tau, \sigma)} \neq 1$, and Σ is not $(X, 2)$ -arc-regular. Recall that X_τ acts faithfully on $\Sigma(\tau)$. It implies $X_{P_\tau} = X_{A_\tau} = X_\tau \cong S_4$. Since $\ell_1 = 1$, we have $X_{(\tau_1, \tau, \sigma, \sigma_1)} = X_{(\tau_1, \tau, \sigma)} \neq 1$. It implies that \mathfrak{J} is not $(X, 1)$ -arc regular.

Let $(\tau, \sigma) \in V(\Xi) = \text{Arc}(\Sigma)$. Set $\Xi((\tau, \sigma)) = \{(\sigma_1, \delta_1), (\sigma_2, \delta_2), (\sigma_3, \delta_3)\}$, the neighborhood of (τ, σ) in Ξ . Then $\Sigma(\tau) = \{\sigma, \sigma_1, \sigma_2, \sigma_3\}$ and $(\sigma, \tau, \sigma_i, \delta_i) \in \Delta$, $i = 1, 2, 3$. It follows from $\ell_1 = 1$ that $\sigma_i^x = \sigma_j$ implies $\delta_i^x = \delta_j$ for $x \in X_{(\tau, \sigma)}$ and $1 \leq i, j \leq 3$. Then $X_{(\tau, \sigma)}$ fixes $\Xi((\tau, \sigma))$ setwise. Since $X_\tau \cong S_4$, we conclude that the permutation group induced by $X_{(\tau, \sigma)}$ on $\Sigma(\tau) \setminus \{\sigma\}$ is isomorphic to S_3 , which is 2-transitive on $\Sigma(\tau) \setminus \{\sigma\}$. Thus $X_{(\tau, \sigma)}$ acts 2-transitively on $\Xi((\tau, \sigma))$. It follows that Ξ is $(X, 2)$ -arc-transitive. Further, checking the number of the 2-arcs of Ξ implies that Ξ is $(X, 2)$ -arc-regular. This completes the proof of (a).

Case 2. $\ell_1 = 2$. In this case, $\text{val}(\mathfrak{J}) = 2\ell_1 = 4$, $\text{val}(\Xi) = 3\ell_1 = 6$ and $\mathfrak{J}[P_\tau, P_\sigma] \cong \Xi[A_\tau, A_\sigma] \cong K_{3,3} - 3K_2$. By Lemma 6.2, both \mathfrak{J} and Ξ are connected.

Now we shall show that X_τ acts faithfully on the neighborhood $\Sigma(\tau)$ of τ in Σ , by a similar argument as in the first paragraph of the proof of Theorem 5.2. Since Σ is $(X, 2)$ -arc transitive, every 2-arc of Σ lies in a member of Δ . Let (τ, σ) be an arbitrary arc of Σ . Since $\ell_1(\Delta) = 2$ and Δ is a self-paired X -orbit, we conclude that, for any $\tau_1 \in \Sigma(\tau) \setminus \{\sigma\}$ there is a unique $\sigma_1 \in \Sigma(\sigma) \setminus \{\tau\}$ such that $(\tau_1, \tau, \sigma, \sigma_1) \notin \Delta$, $X_{(\tau_1, \tau, \sigma)} = X_{(\tau, \sigma, \sigma_1)}$, and that $(\tau'_1, \tau, \sigma, \sigma_1) \notin \Delta$ implies $\tau'_1 = \tau_1$. Then $(X_\tau)_{(\Sigma(\tau))} = \bigcap_{\tau_1 \in \Sigma(\tau) \setminus \{\sigma\}} X_{(\tau_1, \tau, \sigma)} = \bigcap_{\sigma_1 \in \Sigma(\sigma) \setminus \{\tau\}} X_{(\tau, \sigma, \sigma_1)} = (X_\sigma)_{(\Sigma(\sigma))}$. It follows from the connectedness of Σ that $(X_\tau)_{(\Sigma(\tau))}$ fixes every vertex of Σ . Thus $(X_\tau)_{(\Sigma(\tau))} = 1$ and X_τ is faithful on $\Sigma(\tau)$.

For a 2-arc (τ_1, τ, σ) of Σ , since $\ell_1(\Delta) = 2$, there is $\sigma_2, \sigma_3 \in \Sigma(\sigma)$ such that $(\tau_1, \tau, \sigma, \sigma_2) \in \Delta$ and $(\tau_1, \tau, \sigma, \sigma_3) \in \Delta$. Since Δ is an X -orbit, $X_{(\tau_1, \tau, \sigma)}$ acts transitively on $\{\sigma_2, \sigma_3\}$. In particular, $X_{(\tau_1, \tau, \sigma)} \neq 1$. Thus we have

$X_{P_\tau} = X_{A_\tau} = X_\tau \cong S_4$. Further, $|X| = |V(\Sigma)||X_\tau| = 24\mu = |\text{Arc}(\mathfrak{J})| = |\text{Arc}(\Xi)|$, so both \mathfrak{J} and Ξ are $(X, 1)$ -arc-regular.

Set $\Delta' = \text{Arc}_3(\Sigma) \setminus \Delta$. Then Δ' is self-paired and X -invariant. For any two 3-arcs $(\tau_1, \tau, \sigma, \sigma_1)$ and $(\tau'_1, \tau', \sigma', \sigma'_1)$ of Σ in Δ' , since Σ is $(X, 2)$ -arc-transitive, there exists some $x \in X$ such that $(\tau'_1, \tau', \sigma')^x = (\tau_1, \tau, \sigma)$. Then $(\tau_1, \tau, \sigma, \sigma_1^{x'}) = (\tau'_1, \tau', \sigma', \sigma'_1)^x \in \Delta'$. By the argument in the second paragraph of this case, $\sigma_1^{x'} = \sigma_1$, that is, $(\tau'_1, \tau', \sigma', \sigma'_1)^x = (\tau_1, \tau, \sigma, \sigma_1)$. It follows that Δ' is X -transitive and $\ell_1(\Delta') = 1$. Thus (b) holds by Theorem 5.2.

Case 3. $\ell_1(\Delta) = 3$. Then $\text{val}(\mathfrak{J}) = 2\ell_1 = 6$, $\text{val}(\Xi) = 3\ell_1 = 9$ and $\mathfrak{J}[P_\tau, P_\sigma] \cong \Xi[A_\tau, A_\sigma] \cong K_{3,3}$. It follows from [12, Theorem 2] that Σ is $(X, 3)$ -arc transitive. By Lemma 6.2, both \mathfrak{J} and Ξ are connected. Note that $\Sigma \cong \mathfrak{J}_P$ is of valency four. Let σ, τ and δ be three distinct vertices of Σ such that $P_\sigma, P_\delta \in \mathfrak{J}_P(P_\tau)$. Then there exist $\mathfrak{v} \in P_\tau$, $u_1, u_2 \in P_\sigma$ and $\mathfrak{w} \in P_\delta$ such that (u_1, \mathfrak{v}, u_2) and $(\mathfrak{w}, \mathfrak{v}, u_1)$ are 2-arcs of \mathfrak{J} . Since P is X -invariant, there is no $x \in X$ with $(u_1, \mathfrak{v}, u_2)^x = (\mathfrak{w}, \mathfrak{v}, u_1)$. Thus \mathfrak{J} is not $(X, 2)$ -arc-transitive, and so it is $(X, 1)$ -transitive. A similar argument implies that Ξ is $(X, 1)$ -transitive. Hence (c) holds. \blacksquare

Corollary 6.4. *Let Σ be a connected tetravalent $(X, 2)$ -transitive graph. Then either $\Sigma \cong K_5$, or Σ is a near n -gonal graph for some integer $n \geq 4$.*

At the end of this section we give several examples, which indicate that there exist certain graphs satisfying each case listed in Theorem 6.3.

Example 6.5. Let $X = PSL(2, p)$, where p is a prime such that $5 \neq p \equiv \pm 3 \pmod{8}$. Then by [11], there exist $H < X$ and an involution $z \in X$ such that $H \cong A_4$, $P = H \cap H^z =: \langle h \rangle \cong Z_3$, $z \in N_X(P)$ and $h^z = h^{-1}$. Moreover, $\Sigma := \text{Cos}(X, H, HzH) \not\cong K_5$ is a tetravalent $(X, 2)$ -arc-regular graph and $\text{Aut}(\Sigma) = X$. Set $H = P \cup Pg \cup Pg_2 \cup Pg_3$. Let $\Delta = \{(Hzgx, Hx, Hzx, Hzgzx) \mid x \in X\}$. Then Δ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$ with $\mathbf{l}(\Delta) = (1, 1, 1)$.

Example 6.6. Let $X = PSL(2, p)$ for a prime $p \geq 11$ with $p \equiv \pm 1 \pmod{8}$. Let $S_4 \cong H < X$. Then by [13, Lemma 4.1], there exists an involution $z \in X \setminus H$ such that $N_X(P) = P \times \langle z \rangle$, where $P = H \cap H^z \cong S_3$. Further, $\Sigma = \text{Cos}(X, H, HzH)$ is a tetravalent $(X, 2)$ -transitive graph with $\text{Aut}(\Sigma) = X$. Set $H = P \cup Pg \cup Pg_2 \cup Pg_3$. Then $\Delta = \{(Hzgx, Hx, Hzx, Hzgzx) \mid x \in X\}$ is a self-paired X -orbit on $\text{Arc}_3(\Sigma)$ with $\mathbf{l}(\Delta) = (1, 2)$.

Example 6.7. Let $\Sigma = K_{5,5} - 5K_2$ with vertex set $\{i, i' \mid 1 \leq i \leq 5\}$. For $g \in S_5$, define $\bar{g} : i \mapsto g(i)$, $i' \mapsto g(i)'$. Let $z : i \leftrightarrow i'$. Set $X = \langle \bar{g}, z \mid g \in S_5 \rangle$. Then Σ is $(X, 2)$ -transitive. Then both $\Delta_1 := \{(1, 2', 3, 1')^x \mid x \in X\}$ and $\Delta_2 := \{(1, 2', 3, 4')^x \mid x \in X\}$ are self-paired with $\mathbf{l}(\Delta_1) = (1, 2)$ and $\mathbf{l}(\Delta_2) = (2, 1)$.

7. HEPTAVALENT GRAPHS WITH $X_\tau^{\Sigma(\tau)} \cong PSL(3, 2)$

Theorem 7.1. *Let Σ be an $(X, 2)$ -arc-transitive graph of valency 7 with $X_\tau^{\Sigma(\tau)} \cong PSL(3, 2)$ for $\tau \in V(\Sigma)$. Then there exists a self-paired X -symmetric orbit Θ on $DSt^3(\Sigma)$. Let $\Pi = \Pi(\Sigma, \Theta)$ and $\mathcal{S} = St(\Theta)$. Then, for $\sigma \in \Sigma(\tau)$, one of the following cases occurs.*

- (1) $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong 3K_2$, and Π is a trivalent $(X, 2)$ -arc-transitive graph;
- (2) $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong K_{3,3} - 3K_2$, $val(\Pi) = 6$, Π is connected and $(X, 1)$ -transitive;
- (3) $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong K_{3,3}$, $val(\Pi) = 9$ and Π is connected and $(X, 1)$ -transitive.

Proof. Let $\tau \in V(\Sigma)$. Since $X_\tau^{\Sigma(\tau)} \cong PSL(3, 2)$, we may identify $\Sigma(\tau)$ with the point set of seven-point plane $PG(2, 2)$, which is an X_τ -flag-transitive 1-(7, 3, 3) design with multiplicity 1. By Theorem 3.6, there exists a self-paired X -symmetric orbit Θ on $DSt^3(\Sigma)$. Set $\mathcal{S} = St(\Theta)$ and $\Pi = \Pi(\Sigma, \Theta)$. Then, by Theorem 3.5, Π is X -symmetric and $\Pi_{\mathcal{B}} \cong \Sigma$, where $\mathcal{B} = \{\mathcal{S}_\tau \mid \tau \in V(\Sigma)\}$ and $\mathcal{S}_\tau = \{\mathfrak{s} \in \mathcal{S} \mid \mathfrak{s} = \mathfrak{s}(\tau, S), S \subseteq \Sigma(\tau), |S| = 3\}$. Further, for $\mathcal{S}_\tau \in \mathcal{B}$, we have $X_\tau = X_{\mathcal{S}_\tau}$ and $\mathcal{D}(\mathcal{S}_\tau) \cong \mathbb{D}^*(\tau) \cong PG(2, 2)$. (See Section 3 for the definition of $\mathbb{D}(\tau)$.) In particular, for $\sigma \in \Sigma(\tau)$, $|\mathcal{S}_\tau \cap \Pi(\mathcal{S}_\sigma)| = 3$; thus the bipartite graph $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma]$ is isomorphic to one of $3K_2$, $K_{3,3} - 3K_2$ and $K_{3,3}$ as $X_\tau \cap X_\sigma$ acts transitively on the edges of $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma]$. Moreover, noting that, any pair of distinct lines of $PG(2, 2)$ intersect a unique point and any pair of distinct points determine a unique line, it follows that $\lambda := |\Pi(\mathcal{S}_\sigma) \cap \mathcal{S}_\tau \cap \Pi(\mathcal{S}_\delta)| = 1$ for $\sigma, \delta \in \Sigma(\tau)$ with $\sigma \neq \delta$. Then by Lemma 6.2, Π is connected if $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong K_{3,3} - 3K_2$ or $K_{3,3}$. Note that each point of $\mathcal{D}(\mathcal{S}_\tau)$ belongs to three blocks. It follows that Π is of valency 3ℓ , where ℓ is the valency of $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma]$.

Assume first that $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong 3K_2$. Then $val(\Pi) = 3$. Let $\mathfrak{s} \in \mathcal{S}_\tau$, and $\Pi(\mathfrak{s}) = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$ with $\mathfrak{s}_i \in \mathcal{S}_{\tau_i}$ for $i = 1, 2, 3$. Then τ_1, τ_2 and τ_3 are distinct vertices of Σ . Recall $\mathcal{D}(\mathcal{S}_\tau) \cong \mathbb{D}^*(\tau) \cong PG(2, 2)$. Then we may identify \mathfrak{s} with a line L of $PG(2, 2)$, and \mathcal{S}_{τ_i} with the points in this line. Then $(X_\tau^{\Sigma(\tau)})_{\mathfrak{s}} \cong S_4$ acts 2-transitively on $\{\mathcal{S}_{\tau_i} \mid i = 1, 2, 3\}$. It implies that $(X_\tau)_{\mathfrak{s}} = X_{\mathfrak{s}}$ acts 2-transitively (and unfaithfully) on $\{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$. Thus Π is $(X, 2)$ -arc-transitive, and (1) holds.

Now let $\Pi[\mathcal{S}_\tau, \mathcal{S}_\sigma] \cong K_{3,3} - 3K_2$ or $K_{3,3}$. Then Π has two 2-arcs, say $(\mathfrak{v}, \mathfrak{u}, \mathfrak{w})$ and $(\mathfrak{v}', \mathfrak{u}', \mathfrak{w}')$, such that $\mathfrak{v}, \mathfrak{v}', \mathfrak{w} \in \mathcal{S}_\tau$, $\mathfrak{u}, \mathfrak{u}' \in \mathcal{S}_\sigma$ and $\mathfrak{w}' \in \mathcal{S}_\delta$ for distinct τ, σ and δ . Noting \mathcal{B} is X -invariant, there is no $x \in X$ maps $(\mathfrak{v}, \mathfrak{u}, \mathfrak{w})$ to $(\mathfrak{v}', \mathfrak{u}', \mathfrak{w}')$. Thus Π is not $(X, 2)$ -arc-transitive. Then (2) and (3) hold. ■

The following examples indicate that there exist certain graphs satisfying each case listed in Theorem 7.1.

Example 7.2. Let Σ be the complete graph on vectors of \mathbb{F}^3 , where $\mathbb{F} = \{0, 1\}$ is a binary field. Then the 3-dimensional affine group $X := AGL(3, 2)$

is a subgroup of the automorphism group $\text{Aut}(\Sigma) \cong S_8$ of Σ . Set $\mathbf{v}_0 = (0, 0, 0)$, $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0)$, $\mathbf{v}_3 = (1, 1, 0)$, $\mathbf{v}_4 = (0, 0, 1)$, $\mathbf{v}_5 = (1, 0, 1)$, $\mathbf{v}_6 = (0, 1, 1)$ and $\mathbf{v}_7 = (1, 1, 1)$. Then $X_{\mathbf{v}_0} = GL(3, 2) \cong PSL(3, 2)$ is 2-transitive on $\{\mathbf{v}_i \mid i = 1, 2, \dots, 7\}$. Hence Σ is $(X, 2)$ -arc-transitive. We define $t_1 : (a_1, a_2, a_3) \mapsto (a_1, a_2, a_3 + 1)$ and $t_2 : (a_1, a_2, a_3) \mapsto (a_2, a_1, a_3 + 1)$ for $(a_1, a_2, a_3) \in \mathbb{F}^3$, respectively. Then $t_1, t_2 \in X$ with $t_1^2 = t_2^2 = 1$. Let $L = \{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$ and set $\mathfrak{l} = \mathfrak{s}(\mathbf{v}_0, L)$. Note that $\{\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_6\}$ is a subspace of \mathbb{F}^3 . Then $X_{\mathfrak{l}}$ is the stabilizer of this subspace in $GL(3, 2)$. Thus

$$X_{\mathfrak{l}} = \left\{ \begin{bmatrix} 1 & e & f \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \mid \begin{array}{l} a, b, c, d, e, f \in \mathbb{F} \\ ad - bc = 1 \end{array} \right\},$$

$$(X_{\mathfrak{l}})_{\mathbf{v}_4} = \left\{ \begin{bmatrix} 1 & e & f \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid b, e, f \in \mathbb{F} \right\}.$$

Let $\mathbf{r}_i = \mathfrak{s}(\mathbf{v}_4, L^{t_i})$ for $i = 1$ and 2 . Then $L^{t_1} = \{\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_6\}$, $L^{t_2} = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_5\}$, $\mathfrak{l}^{t_i} = \mathbf{r}_i$ and $\mathbf{r}_i^{t_i} = \mathfrak{l}$. Thus $\Theta_i := \{(\mathfrak{l}^x, \mathbf{r}_i^x) \mid x \in X\}$ is a self-paired X -orbits on $DSt^3(\Sigma)$. Let $\Pi^i = \Pi(\Sigma, \Theta_i)$ and $\Delta_i = \Pi^i[\mathcal{S}_{\mathbf{v}_0}, \mathcal{S}_{\mathbf{v}_4}]$ for $i = 1$ and 2 . Note that $X_{\mathcal{S}_{\mathbf{v}_0}} \cap X_{\mathcal{S}_{\mathbf{v}_4}} = X_{\mathbf{v}_0} \cap X_{\mathbf{v}_4}$ acts transitively on the edges of Δ_i . It follows that $(X_{\mathbf{v}_0} \cap X_{\mathbf{v}_4})_{\mathfrak{l}} = (X_{\mathfrak{l}})_{\mathbf{v}_4}$ is transitive on the neighborhood of \mathfrak{l} in Δ_i . Thus $\text{val}(\Delta_i) = |\{\mathbf{r}_i^x \mid x \in (X_{\mathfrak{l}})_{\mathbf{v}_4}\}|$. If $i = 1$, then $\mathbf{r}_1^x = \mathfrak{s}(\mathbf{v}_4, L^{t_1 x}) = \mathfrak{s}(\mathbf{v}_4, L^{t_1}) = \mathbf{r}_1$ for $x \in (X_{\mathfrak{l}})_{\mathbf{v}_4}$, so $\text{val}(\Delta_1) = 1$ and Theorem 7.1 (1) occurs. (In fact, $\Pi_1 \cong 14K_4$. We omit the detail.) If $i = 2$, then $L^{t_2 x} = L^{t_2}$ or $\{\mathbf{v}_0, \mathbf{v}_3, \mathbf{v}_7\}$ for $x \in (X_{\mathfrak{l}})_{\mathbf{v}_4}$, thus $\text{val}(\Delta_2) = 2$ and Theorem 7.1 (2) occurs.

Example 7.3. Let $\mathbb{F} = \{0, 1\}$ be a binary field. Denote by \mathbf{i} the non-zero vector of \mathbb{F}^3 with coordinate (a_1, a_2, a_3) such that $i = 4a_1 + 2a_2 + a_3$. Let Σ be the complete bipartite graph with vertex set $\{\mathfrak{l}\mathbf{i} \mid 1 \leq i \leq 7\} \cup \{\mathbf{r}\mathbf{i} \mid 1 \leq i \leq 7\}$. Then $X := PSL(3, 2) \wr Z_2$ is a subgroup of $\text{Aut}(\Sigma)$, and Σ is $(X, 3)$ -transitive. Let $L = \{\mathbf{r}\mathbf{1}, \mathbf{r}\mathbf{2}, \mathbf{r}\mathbf{3}\}$ and $R = \{\mathfrak{l}\mathbf{1}, \mathfrak{l}\mathbf{2}, \mathfrak{l}\mathbf{3}\}$. Set $\mathfrak{l} = \mathfrak{s}(\mathfrak{l}\mathbf{1}, L)$, $\mathbf{r} = \mathfrak{s}(\mathbf{r}\mathbf{1}, R)$ and $\Theta_3 := \{(\mathfrak{l}, \mathbf{r})^x \mid x \in X\}$. Then Θ_3 is a self-paired X -symmetric orbit on $DSt^3(\Sigma)$, and $\Pi := \Pi(\Sigma, \Theta)$ satisfies Theorem 7.1 (3).

8. PROOF OF THEOREM 4.1

Now we are ready to give the proof of Theorem 4.1.

Since Γ is X -symmetric and $\Gamma_{\mathcal{B}}$ contains at least one edge, $\Gamma_{\mathcal{B}}$ is X -symmetric, that is, $X_{\mathcal{B}}$ is transitive on $\Gamma_{\mathcal{B}}(B)$ for $B \in \mathcal{B}$; further, B is an independent subset of $V(\Gamma)$.

We first show that each of Theorem 4.1(a)-(d) implies the $(X, 2)$ -arc-transitivity of $\Gamma_{\mathcal{B}}$. It suffices to show that $X_{\mathcal{B}}$ acts 2-transitively on $\Gamma_{\mathcal{B}}(B)$ for $B \in \mathcal{B}$. It is trivial for the case (d) as $\Gamma_{\mathcal{B}}(B)$ is the block set of $\mathcal{D}(B)$. In the following we assume one of (a), (b) and (c) occurs.

Suppose that $m := m(\mathcal{D}(B)) \neq 1$. Then $\Gamma_{\mathcal{B}}(B)$ admits an X_B -invariant partition $\mathcal{M} := \{\mathcal{M}_C \mid C \in \Gamma_{\mathcal{B}}(B)\}$, where \mathcal{M}_C is a set of blocks of $\mathcal{D}(B)$ with the same trace $B \cap \Gamma(C)$ of C . Thus $m = |\mathcal{M}_C|$ is a divisor of b . For $\mathfrak{v} \in B$, it is easily to see that $C \in \Gamma_{\mathcal{B}}(\mathfrak{v})$ yields $D \in \Gamma_{\mathcal{B}}(\mathfrak{v})$ for any $D \in \mathcal{M}_C$. This observation says that $m = |\mathcal{M}_C|$ is a divisor of $r := |\Gamma_{\mathcal{B}}(\mathfrak{v})|$. It follows that $(v, b, r) = (6, 4, 2)$, $m = 2 = r$ and $|\mathcal{M}| = 2$. Set $\mathcal{M} = \{\mathcal{M}_C, \mathcal{M}_D\}$. Then $\mathcal{T} := \{B \cap \Gamma(C), B \cap \Gamma(D)\}$ is an X_B -invariant partition of B . Let K be the kernel of X_B acting on \mathcal{T} . Then $|X_B : K| = 2$ and $X_{(B)} \leq K$. It follows that $X_B^B \cong S_4$ and $K/X_{(B)} \cong A_4$. Note that K is in fact the set-wise stabilizer of $B \cap \Gamma(C)$, and also of $B \cap \Gamma(D)$, in X_B . Then K is transitive on $B \cap \Gamma(C)$ and on $B \cap \Gamma(D)$. Let H and H_1 be the kernels of K acting on $B \cap \Gamma(C)$ and on $B \cap \Gamma(D)$, respectively. Then K/H and K/H_1 are permutation groups of degree 3. Noting that $X_{(B)} \leq H$ and $X_{(B)} \leq H_1$, it follows that $H/X_{(B)}$ and $H_1/X_{(B)}$ are normal subgroups of $K/X_{(B)}$ with index 3 in $K/X_{(B)}$. Hence $H_1/X_{(B)} = H/X_{(B)}$ as A_4 has only one normal subgroup of order 4. Thus $H_1 = H$ fixes B point-wise, and so $H \leq X_{(B)}$, which contradicts $|H/X_{(B)}| = 4$.

Suppose that $m^*(\Gamma, \mathcal{B}) \neq 1$. Recall that $m^*(\Gamma, \mathcal{B}) := |B \cap (\cap_{C \in \Gamma_{\mathcal{B}}(\mathfrak{v})} \Gamma(C))|$, the multiplicity of the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$, is independent of the choices of B and $\mathfrak{v} \in B$. Assume that $\mathcal{D}^*(B)$ is a 1 -(v^*, b^*, r^*) design. Then $(v^*, b^*, r^*) = (b, v, k)$ is one of $(4, 4, 3)$, $(4, 6, 3)$ and $(7, 7, 3)$. A similar argument as in the above paragraph implies that $m^*(\Gamma, \mathcal{B})$ is a divisor of v and of k . Then $(b, v, k) = (4, 6, 3)$ and $m^*(\Gamma, \mathcal{B}) = 3 = k$. It follows that $m(\mathcal{D}(B)) \geq |\Gamma_{\mathcal{B}}(\mathfrak{v})| = 2$, again a contradiction.

The above argument gives $m(\mathcal{D}(B)) = 1$ and $m^*(\Gamma, \mathcal{B}) = 1$. Then $X_B^{\Gamma_{\mathcal{B}}(B)} \cong X_B^B$ by Theorem 3.8. Thus $X_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive on $\Gamma_{\mathcal{B}}(B)$ if one of cases (a), (b) and (c) occurs. Therefore, if one of Theorem 4.1(a)-(d) occurs, then X_B acts 2-transitively on the blocks of $\mathcal{D}(B) = \Gamma_{\mathcal{B}}(B)$, and hence $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive.

Now assume that $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive. Recall that $m(\mathcal{D}(B))$ is the multiplicity of $\mathcal{D}(B)$, the number $C \in \Gamma_{\mathcal{B}}(B)$ with the same trace, which is independent of the choice of B . Then $m(\mathcal{D}(B)) = 1$ by [14, Lemma 2.4]. Since $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive, $\lambda := |\Gamma(C) \cap B \cap \Gamma(D)|$ is independent of the choice of $[C, B, D] \in \text{Path}_2(\Gamma_{\mathcal{B}})$. By [14, Corollary 3.3], $vr = 3b$ and $\lambda(b-1) = 3(r-1)$, thus $(9-\lambda v)r = 3(3-\lambda)$. Since Γ is not a multicover of $\Gamma_{\mathcal{B}}$, we have $\lambda \leq k-1 = 2$ and $v > k$. If $\lambda = 0$, then $r = 1$ and $v = 3b$. Let $\lambda \geq 1$. Then, by [14, Theorem 3.2], the dual design $\mathcal{D}^*(B)$ of $\mathcal{D}(B)$ is a 2 -(b, r, λ) design with v blocks. The well-known Fisher's Inequality applied to $\mathcal{D}^*(B)$ gives $b \leq v$, and so $r \leq k = 3$. If $\lambda = 2$, then $\lambda(b-1) = 3(r-1)$, $(9-2v)r = 3$ and $v > k$ imply $(v, b, r) = (4, 4, 3)$. If $\lambda = 1$, then $r \leq k$, $vr = 3b$ and $(9-v)r = 6$ yield $(v, b, r) = (6, 4, 2)$ or $(7, 7, 3)$.

Note that $1 \leq m^*(\Gamma, \mathcal{B}) \leq \lambda$ if $\lambda \neq 0$. Suppose that $m^*(\Gamma, \mathcal{B}) \neq 1$ for some $\lambda \neq 0$. Then $\lambda = 2 = m^*(\Gamma, \mathcal{B})$. It follows from $r = 3 = k$ that there are $C, D \in \Gamma_{\mathcal{B}}(\mathfrak{v})$ such that $C \neq D$ and $B \cap \Gamma(C) = B \cap \Gamma(D)$. Thus C and D has the same trace, and hence $m(\mathcal{D}(B)) \geq 2$, a contradiction. Therefore, if $\lambda \neq 0$ then $m^*(\Gamma, \mathcal{B}) = 1$ and, by Theorem 3.7 and 3.8, $X_B^{\Gamma_{\mathcal{B}}(B)} \cong X_B^B$, and the induced action of X on \mathcal{B} is faithful.

We treat four separate cases in the following.

Case 1. $(v, b, r, \lambda) = (4, 4, 3, 2)$. Then $\text{val}(\Gamma_{\mathcal{B}}) = 4$, and $X_B^B \cong A_4$ or S_4 as X_B acts 2-transitively on $\Gamma_{\mathcal{B}}(B)$. Thus (a) holds.

By [12, Theorem 2], $\Gamma \cong \Xi(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired X -orbit Δ on $\text{Arc}_3(\Gamma_{\mathcal{B}})$. For any connected tetravalent $(X, 2)$ -arc-transitive graph Σ , by Theorem 5.1, there exists some self-paired X -orbit on $\text{Arc}_3(\Sigma)$, and by [12, Theorem 10], the corresponding 3-arc graph admits an X -invariant partition with quotient graph isomorphic to Σ and parameters $(v, b, k, r) = (4, 4, 3, 3)$. Thus, by Theorem 6.3, one of (a.1), (a.2) and (a.3) of Theorem 4.1 holds.

Case 2. $(v, b, r, \lambda) = (6, 4, 2, 1)$. Then $\text{val}(\Gamma_{\mathcal{B}}) = 4$, $X_B^B \cong A_4$ or S_4 , and so (b) occurs.

Since $(r, \lambda) = (2, 1)$, by Lemma 2.2, $\Gamma \cong \mathbb{J}(\Gamma_{\mathcal{B}}, \Delta)$ for some self-paired X -orbit Δ on $\text{Arc}_3(\Gamma_{\mathcal{B}})$. By Theorem 6.1, $J(\Delta)$ is a self-paired X -orbit on $J(\Gamma_{\mathcal{B}})$, and $\mathbb{J}(\Gamma_{\mathcal{B}}, \Delta) \cong \Psi(\Gamma_{\mathcal{B}}, J(\Delta))$. For any connected tetravalent $(X, 2)$ -arc-transitive graph Σ , by Theorem 5.1, there exists some self-paired X -orbit on $\text{Arc}_3(\Sigma)$, and by Proposition 2.1 and Theorem 6.3, the corresponding graph constructed as in Proposition 2.1 admits an X -invariant partition with quotient graph isomorphic to Σ and parameters $(v, b, r, \lambda) = (6, 4, 2, 1)$. Then (b.1), (b.2) or (b.3) follows from Theorem 6.3,

Case 3. $(v, b, r, \lambda) = (7, 7, 3, 1)$. In this case, $\mathcal{D}(B) \cong PG(2, 2)$ is X_B -flag-transitive. Then $X_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a subgroup of $PSL(3, 2)$, the automorphism group of $PG(2, 2)$. Since $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive, $X_B^{\Gamma_{\mathcal{B}}(B)}$ is 2-transitive on $\Gamma_{\mathcal{B}}(B)$, and hence $|X_B^{\Gamma_{\mathcal{B}}(B)}| \geq 42$. It follows that $X_B^{\Gamma_{\mathcal{B}}(B)} \cong PSL(3, 2)$. Thus $X_B^B \cong X_B^{\Gamma_{\mathcal{B}}(B)} \cong PSL(3, 2)$ by Theorem 3.8. Hence (c) holds. Since $m^*(\Gamma, \mathcal{B}) = 1$, by Theorem 3.7, $\Gamma \cong \Pi(\Gamma_{\mathcal{B}}, \Theta)$ for some self-paired X -symmetric orbit Θ on $DSt^3(\Gamma_{\mathcal{B}})$. Further, by Theorem 7.1 and the above argument, one connected heptavalent $(X, 2)$ -arc-transitive graph Σ occurs as $\Gamma_{\mathcal{B}}$ if and only if $X_{\tau}^{\Sigma(\tau)} \cong PSL(3, 2)$. Again by Theorem 7.1, one of (c.1), (c.2) and (c.3) holds.

Case 4. $\lambda = 0$, $r = 1$ and $v = 3b$. Since $\Gamma_{\mathcal{B}}$ is $(X, 2)$ -arc-transitive, X_B acts 2-transitively on the blocks of $\mathcal{D}(B)$. It follows from $r = 1$ and $\lambda = 0$ that $\Gamma \cong e\Gamma[B, C]$ for $\{B, C\} \in E(\Gamma_{\mathcal{B}})$. Since $k = 3$, we know $\Gamma[B, C] \cong 3K_2, K_{3,3} - 3K_2$ or $K_{3,3}$, so one of (d.1), (d.2) and (d.3) occurs.

REFERENCES

- [1] N.L. Biggs, Algebraic graph theory, Second Edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1993.
- [2] T. Beth, D. Jungnickel and H. Lenz, Design theory, Second Edition, Cambridge University Press, 1999.
- [3] J.A. Bondy and U.S.R. Murty, Graph theory with applications, The Macmillan Press, London and Basingstoke, 1976.
- [4] J.D. Dixon and B. Mortimer, Permutation groups, Springer, New York, 1996.
- [5] X.G. Fang and C.E. Praeger, Finite two-arc transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27**(1999), 3755-3769.
- [6] X.G. Fang and C.E. Praeger, Finite two-arc transitive graphs admitting a Ree simple group, *Comm. Algebra* **27**(1999), 3727-3754.
- [7] A. Gardiner and C.E. Praeger, A geometrical approach to imprimitive graphs, *Proc. London Math. Soc.* (3) **71**(1995), 524-546.
- [8] A. Gardiner and C.E. Praeger, Topological covers of complete graphs, *Math. Proc. Cambridge Philos. Soc.* **123**(1998), 549-559.
- [9] A. Gardiner and C.E. Praeger, Symmetric graphs with complete quotients, preprint.
- [10] M.A. Iranmanesh, C.E. Praeger and S. Zhou, Finite symmetric graphs with two-arc-transitive quotients, *J. Combin Theory B* **94**(2005), 79-99.
- [11] C.H. Li, Finite s-arc-transitive graphs of odd order, *J. Combin. Theory Ser. B* **81**(2001), 307-317.
- [12] C.H. Li, C.E. Praeger and S. Zhou, A class of finite symmetric graphs with 2-arc-transitive quotients, *Math. Proc. Cambridge Phil. Soc.* **129**(2000), 19-34.
- [13] C.H. Li, Z.P. Lu and D. Marušič, On primitive permutation groups with small sub-orbits and their orbital graphs, *J. Algebra* **279**(2004), 749-770.
- [14] Z.P. Lu and S. Zhou, Finite Symmetric Graphs with Two-Arc Transitive Quotients II, *J. of Graph Theory* **56**(2007), 167-193.
- [15] M. Perkel, Near-polygonal graphs, *Ars Combin.* **26**(A)(1988), 149-170.
- [16] S. Zhou, Imprimitive symmetric graphs, 3-arc graphs and 1-designs, *Discrete Math.* **244**(2002), 521-537.
- [17] S. Zhou, Constructing a class of symmetric graphs, *Eur. J. Combin.* **23** (2002), 741-760.
- [18] S. Zhou, Almost covers of 2-arc-transitive graphs, *Combinatorica* **24**(2004), 731-745.
- [19] S. Zhou, A local analysis of imprimitive symmetric graphs, *J Algebraic Combin.* **22**(2005), 435-449.
- [20] S. Zhou, On a class of finite symmetric graphs, *Eur. J. Combina.* **29**(2008), 630-640.
- [21] S. Zhou, Classification of a family of symmetric graphs with complete quotients, *Discrete Math.*, accepted.

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